A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain** of T, and the set $\mathbb{R}^m$ is called the **codomain** of T. The notation $T:\mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain of T is $\mathbb{R}^m$. For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ under T. The set of all images $T(\mathbf{x})$ is called the range of T.

Let $\mathbf{u}$ and $\mathbf{v}$ be in the domain of T, and let c be any scalar. The transformation T is **linear** if and only if (1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and (2) $T(c\mathbf{u}) = cT(\mathbf{u})$.

If T is a linear transformation, then (1) $T(\mathbf{0}) = \mathbf{0}$ and (2) $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.

Theorem Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. There exists a unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. The matrix A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$.

This matrix A is called the **standard matrix for the linear transformation** T.

Definition A mapping $T:\mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.

That is, the range of T is all of the codomain $\mathbb{R}^m$. That is, T is onto $\mathbb{R}^m$ if, for each $\mathbf{b}$ in $\mathbb{R}^m$, there is at least one solution to $T(\mathbf{x}) = \mathbf{b}$. The mapping T is not onto when there is some $\mathbf{b}$ in $\mathbb{R}^m$ for which the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution.

Definition A mapping $T:\mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.

That is, T is one-to-one if, for each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $T(\mathbf{x}) = \mathbf{b}$ has either a unique solution or no solution. The mapping T is not one-to-one when some $\mathbf{b}$ in $\mathbb{R}^m$ is the image of more than one vector in $\mathbb{R}^n$.

Theorem Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. T is one-to-one if and only if the equation $T(\mathbf{x})=\mathbf{0}$ has only the trivial solution.

Note that the equations $T(\mathbf{x})=\mathbf{0}$ and $A\mathbf{x}=\mathbf{0}$ are the same except for notation. The theorem above then states that T is one-to-one if and only if the equation $A\mathbf{x}=\mathbf{0}$ has only the trivial solution. This is the case if and only if the columns of A are linearly independent.

Theorem Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then (1) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$ and (2) T is one-to-one if and only if the columns of A are linearly independent.

Example Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Determine if T is a one-to-one linear transformation. Determine if T maps $\mathbb{R}^2$ onto $\mathbb{R}^3$.