

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in \mathbb{R}^n and given scalars c_1, c_2, \dots, c_p , the vector \mathbf{y} defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$$

is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ with **weights** c_1, c_2, \dots, c_p .

A vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_p\mathbf{a}_p = \mathbf{b}$ has the same solution set as the linear system whose augmented matrix is $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_p & \mathbf{b} \end{bmatrix}$. In particular, \mathbf{b} can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p$ if and only if there is a solution to the linear system corresponding to $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_p & \mathbf{b} \end{bmatrix}$.

If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in \mathbb{R}^n , then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ is called the **subset of \mathbb{R}^n spanned** (or **generated**) **by** $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$, denoted $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$.

That is, a vector \mathbf{b} is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ if \mathbf{b} is a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$.

Consider the vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^3 such that \mathbf{u} is not a multiple of \mathbf{v} . $\text{Span}\{\mathbf{v}\}$ is the set of scalar multiples of \mathbf{v} , and can be visualized as the set of points in \mathbb{R}^3 on the line through \mathbf{v} and $\mathbf{0}$. $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is the plane in \mathbb{R}^3 that contains \mathbf{u} , \mathbf{v} and $\mathbf{0}$.

If A is an $m \times n$ matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ and if \mathbf{x} is in \mathbb{R}^n , the **product** of A and \mathbf{x} , denoted $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries of \mathbf{x} as weights. That is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n.$$

Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in \mathbf{x} .

If A is an $m \times n$ with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ and if \mathbf{b} is in \mathbb{R}^m , the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution as the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{b}$ which has the same solution set as the linear system whose augmented matrix is $\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$.

Let A be an $m \times n$ matrix. The following are logically equivalent. That is, for a particular matrix A , they are either all true or all false.

1. For each \mathbf{b} in \mathbb{R}^m , the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. Each \mathbf{b} in \mathbb{R}^m is a linear combination of the columns of A .
3. The columns of A span \mathbb{R}^m .
4. A has a pivot position in every row.