Given vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, ..., c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_p\mathbf{v_p}$$

is called a **linear combination** of the vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}$ with **weights** $c_1, c_2, ..., c_p$.

A vector equation $x_1\mathbf{a_1} + x_2\mathbf{a_2} + ... + x_p\mathbf{a_p} = \mathbf{b}$ has the same solution set as the linear system whose augmented matrix is $\begin{bmatrix}\mathbf{a_1} & \mathbf{a_2} & ... & \mathbf{a_p} & \mathbf{b}\end{bmatrix}$. In particular, **b** can be generated by a linear combination of $\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_p}$ if and only if there is a solution to the linear system corresponding to $\begin{bmatrix}\mathbf{a_1} & \mathbf{a_2} & ... & \mathbf{a_p} & \mathbf{b}\end{bmatrix}$.

If $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}$ in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}$ is called the **subset of** $\mathbb{R}^n$ **spanned** (or **generated**) **by** $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}$, denoted $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}\}$.

That is, a vector **b** is in $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}\}$ if **b** is a linear combination of vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}$.

Consider the vectors **u** and **v** in $\mathbb{R}^3$ such that **u** is not a multiple of **v**. $\text{Span}\{\mathbf{v}\}$ is the set of scalar multiples of **v**, and can be visualized as the set of points in $\mathbb{R}^3$ on the line through **v** and **0**. $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ that contains **u**, **v** and **0**.

If A is an $m \times n$ matrix with columns $\mathbf{a_1}, ..., \mathbf{a_n}$ and if **x** is in $\mathbb{R}^n$, the **product** of A and **x**, denoted $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries of **x** as weights. That is,

$$A\mathbf{x} = \begin{bmatrix}\mathbf{a_1} & \mathbf{a_2} & ... & \mathbf{a_n}\end{bmatrix}\begin{bmatrix}x_1 \\ \vdots \\ x_n\end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ...x_n\mathbf{a}_n.$$

Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in **x**.

If A is an $m \times n$ with columns $\mathbf{a_1}, ..., \mathbf{a_n}$ and if **b** is in $\mathbb{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution as the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ...x_n\mathbf{a}_n = \mathbf{b}$ which has the same solution set as the linear system whose augmented matrix is $\begin{bmatrix}\mathbf{a_1} & \dots & \mathbf{a}_n & \mathbf{b}\end{bmatrix}$.

Let A be an $m \times n$ matrix. The following are logically equivalent. That is, for a particular matrix A, they are either all true or all false.

1. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. The columns of A span $\mathbb{R}^m$.
4. A has a pivot position in every row.