Properties of Real Numbers
Especially Distributive Property

Complete this worksheet and hand it in on Friday (the day of the first exam). It will count the same as one homework assignment.

The order of operations that we study in section 1.6 dictates what order is implied by the symbols in a problem. Often, it is more convenient to do things in a different order. You must learn how this is done and when changing things gives the same result. Our goal is to make a problem easier to do while still getting the correct answer!

For example, the order of operations says that if the only operations in your problem are subtraction and addition, the order that is implied is from left to right.

$-3 + 12 - 5$
$= 9 - 5$ since $-3 + 12$ is 9
$= 4$

As we have worked on earlier however, if the only operation is addition, the order that you add the numbers together can be changed without changing the result. For example,

$-3 + 5 + (-7)$
$= 2 + (-7)$
$= -5$

can be done in the following way if you would like, since it's all addition and you are only changing the order:

$-3 + 5 + (-7)$
$= 5 + (-3) + (-7)$
$= 5 + (-10)$
$= -5$

Adding the -3 to the -7 first is fine!

In general, $a + b = b + a$ and $(a + b) + c = a + (b + c)$, that is, neither the order that you write the sum nor the order that you do the sum matters. You get the same result!

However, the order that you subtract numbers together DOES change the result, or at least it changes the sign of the result. That is, $a - b = -(b - a)$. If you want to change the order, first covert your subtraction to adding the opposite. That is, make $a - b$ into $a + (-b)$ and you can change the order!

Exercises: For the following 4 exercises, first perform the operations as written. Then, correctly change the order and show that the result is the same. If it's subtraction, covert to adding the opposite before changing the order.

1. $3 + (-5)$
2. $-6 + 12 + (-4)$
3. $5 - 7$
4. $\frac{3}{5} + \frac{7}{4} - \frac{2}{5} - \frac{3}{4}$

Multiplication and division have the same order properties as addition and subtraction. That is, order of operations says that if all you have is division and multiplication, the order that is implied is from left to right.

$$\begin{aligned} &-12 \div 3 \cdot (-2) \\ &\quad = -4 \cdot (-2) \\ &\quad = 8 \end{aligned}$$

If you tried to do the 3 times -2 first, you would get a final result of 2 which is incorrect.

However, if the only operations are multiplication, changing the order doesn't change the result. For example, let's say you had $\frac{1}{2} \cdot (-5) \cdot 13 \cdot 2$. As written, this problem is a pain! The sign of the answer is no problem; one negative so the answer is negative. But the rest is no fun.

$$\begin{aligned} &\tfrac{1}{2} \cdot (-5) \cdot 13 \cdot 2 \\ &\qquad = -\left(\tfrac{1}{2} \cdot (5) \cdot 13 \cdot 2\right) \\ &\qquad = -\left(\tfrac{5}{2} \cdot 13 \cdot 2\right) \\ &\qquad = -\left(\tfrac{65}{2} \cdot 2\right) \\ &\qquad = -65 \end{aligned}$$

Instead, seeing the 5 times 2 which is 10, and knowing how easy 10 is to use in multiplication, I would choose to do it in the following way:

$$\begin{aligned} &\tfrac{1}{2} \cdot (-5) \cdot 13 \cdot 2 \\ &\qquad = -5 \cdot 2 \cdot 13 \cdot \tfrac{1}{2} \\ &\qquad = -10 \cdot 13 \cdot \tfrac{1}{2} \\ &\qquad = -130 \cdot \tfrac{1}{2} \\ &\qquad = -65 \end{aligned}$$

and the only hard part is knowing that half of 130 is 65.

Unfortunately, the order that you perform divisions DOES matter. If you want to change the order, you must convert division to multiplication by the reciprocal. That is, make $a \div b$ into $a \cdot \frac{1}{b}$.

Exercises: For the following 3 exercises, first perform the operations as written. Then, correctly change the order and show that the result is the same. If it's division, covert to multiplying by the reciprocal before changing the order.

1. $-2 \cdot 7 \cdot 5 \cdot (-1)$

2. $\frac{1}{3} \cdot \frac{5}{7} \cdot \frac{9}{10} \cdot \frac{14}{3}$

3. $7 \div (-2) \cdot 4$

The number 1 has properties that make it very useful, especially when dealing with fractions. Two of the most powerful properties with the number one are:

1 times any number equals that number. That is, $1 \cdot a = a \cdot 1 = a$

and

any number (except 0) divided by itself equals 1. That is, $\frac{a}{a} = 1$, as long as $a \neq 0$.

We know that in a fraction problem, the answer must be reduced to lowest terms. These properties, and the way fractions are multiplied, are how this is done! For example,

$\frac{6}{8} = \frac{2 \cdot 3}{2 \cdot 4} = \frac{2}{2} \cdot \frac{3}{4}$ by fraction multiplication

$= 1 \cdot \frac{3}{4}$ since $\frac{a}{a} = 1$

$= \frac{3}{4}$ since $1 \cdot a = a$

Also, we know that we need a common denominator when adding or subtracting fractions. The properties of 1 are used to convert given fractions to equivalent fractions with the same denominator. For example, let's say we are given the problem $\frac{3}{4} + \frac{5}{6}$. First, we look for the Least Common Multiple of 4 and 6. A list of the multiples of each number will help:

$1 \cdot 4 = 4$, $2 \cdot 4 = 8$, $3 \cdot 4 = 12$, $4 \cdot 4 = 16$, etc.

$1 \cdot 6 = 6$, $2 \cdot 6 = 12$, $3 \cdot 6 = 18$, $4 \cdot 6 = 24$, etc.

12 is the smallest number that is a multiple of each, so we will use that. First,

$\frac{3}{4} = 1 \cdot \frac{3}{4}$ since $1 \cdot a = a$

$= \frac{3}{3} \cdot \frac{3}{4}$ since $1 = \frac{a}{a}$

$= \frac{9}{12}$ by fraction multiplication

and $\frac{5}{6} = 1 \cdot \frac{5}{6} = \frac{2}{2} \cdot \frac{5}{6} = \frac{10}{12}$

so, $\frac{3}{4} + \frac{5}{6} = \frac{9}{12} + \frac{10}{12} = \frac{19}{12}$ which is already in lowest terms since 19 and 12 have no common factors.

Exercises: Perform the following fraction exercises showing all of your steps:

1. $\frac{1}{6} - \frac{3}{8}$
2. $\frac{5}{8} \div \left(-\frac{15}{4}\right)$
3. $\frac{1}{14} - \frac{1}{21} + \frac{5}{6}$
4. Reduce $\frac{60}{126}$ to lowest terms.

I have saved the best property for last. On the surface it is no big deal, but as you progress through this class and beyond, it is probably the most widely used property in algebra. It is also the only property that combines the operations of addition and multiplication.

The Distributive Property of Multiplication Over Addition is difficult to say in English. I will try, and then will have some examples and many exercises so that you can practice what it means. More or less it says that a number multiplied by a sum is the same as the sum of two products. The outside number times the first addend in the sum plus the outside number times the second addend. (I told you it was hard to say in English!! Those of you who speak another language know that there are just some things that don't translate well!!) Here it is in Mathese:

$a \cdot (b + c) = a \cdot b + a \cdot c$

and it works this way too:

$(b + c) \cdot a = b \cdot a + c \cdot a$

The deal is that Order of Operations says the sum inside the parenthesis goes first before multiplying by the outside number. Distributive Property says if you multiply b and c each by a first and add the two products after, you will get the same result. Here's an example. Let's say you have $-2 \cdot (-5 + 13)$. By Order of Operations, your first step is the sum giving $-2 \cdot 8$ which is -16. Distributive Property gives you an alternative. Here is the same problem done using Distributive Property:

$$\begin{aligned} -2 \cdot (-5 + 13) &= (-2)(-5) + (-2)(13) \\ &= 10 + (-26) \\ &= -16 \end{aligned}$$

You can even distribute a negative sign!

$-(12 - 5) = -1 \cdot (12 + (-5))$

$= (-1)(12) + (-1)(-5)$

$= -12 + 5 = -7$

One last thing before some exercises. We are about to start using variables, that is, letters that stand for unknown numbers. With variables, like with parentheses, it is not necessary to write the symbol of multiplication. For example, $5 \cdot x$ can be written $5x$ just like $5 \cdot (2 + 4)$ can be written $5(2 + 4)$ or even $5 \cdot 2$ can be written $(5)(2)$. This gives distributive property a new twist. $5(x + 2)$ cannot be done at all with order of operations! If x is an unknown number, how do we know how much $x + 2$ is? With distributive property we can change $5(x + 2)$ to $5x + 5 \cdot 2$ which is $5x + 10$. Not much, but it's something.

Exercises: In the first 3 exercises, do each problem twice. First, following order or operations, then using distributive property. Notice that the result is the same. For the next 7 exercises, notice that order of operations doesn't get you very far because of the variables, then simplify using distributive property.

1. $-5 \cdot (-3 + (-8))$
2. $(100 - 3) \cdot 5$ (Did you know 97 times 5 was so easy?!?)
3. $2(4 - 6 - 8 + 10)$ (Dist. Prop. doesn't always make it easier...it just gives you a choice.)
4. $3(x + 5)$
5. $3(2x + 5)$
6. $-2(x - 1)$
7. $-4(2x + 8y)$
8. $-2 - 6(2x + 3)$ Careful! Dist. Prop. only applies to part of this!
9. $8 - (x + 3)$ Here too.
10. $\frac{1}{2}(4x + 8)$