1 Numbers

1.1 Define Numbers

Numbers have two main uses in our world: to identify something the same way that a name does, and to represent a value. Some examples of identification numbers are account numbers, phone numbers, address numbers, social security and G-numbers, and titles or names like Chapter 5 or R2D2. Numbers that represent value are much more common, and are the subject of the study of mathematics. Some examples are anything to do with money like prices or wages, scores in sports or on an exam, measurements like area, distance, weight or speed, a time or date, and age. Often, value numbers are used to compare one object with another. For example, a jacket which costs $45 with another that costs $100. A job that offers a salary of $25,000 per year with a job that offers a salary of $40,000 per year. An apartment with area of 500 square feet with one with area 800 square feet. People use value numbers to make decisions, and mathematics helps people make informed decisions.

1.2 Do numbers have meaning if we don't associate them with "real life"?

Exercise 1-1: 10, 16, 0, 21, 5

Discuss what each number in the list above reminds you of. Notice that numbers just by themselves are a very abstract concept. When numbers are associated with a unit they are more meaningful. Therefore Units are associated with numbers. For Example: 10 *yards*, 16 *trees*, 0 *dollars*, 21 *years old*, 5 *students*

Exercise 1-2:

Find any numbers from a newspaper. Categorize them as identification numbers, or value numbers. For the value numbers write a sentence explaining what object in real life is being represented with that value and include the units if there are any. Discuss.

1.3 Comparing Numbers: What determines if one number is greater than another?

Exercise 1-3: To search for the answer to this question let's do the following exercise:

We use the symbol "$>$" to indicate when one number represents more of something than another number. For instance, since 8 pounds of something is more than 5 pounds, we would write 8 lbs. $>$ 5 lbs. We use "$<$" to indicate when one number represents less of something than another number. From the same example above we would write 5 lbs. $<$ 8 lbs.

1. For each question insert either $>$, $<$, or a ? if it is unclear whether one quantity is more or less than the other.
   (a) 10 dogs ☐ 7 dogs
   (b) 10 dogs ☐ 7 cats
   (c) 10 dogs ☐ 7 buckets
   (d) $5 ☐ 34 ¢
   (e) 2 feet ☐ 20 inches
   (f) 12 miles ☐ 3 gallons
2. Which part(s) of question (1) could you clearly answer? Why?
3. What information do you need to answer the other questions? Why?
4. Write a conclusion about what you need in order to compare two things using $>$ or $<$.

Activity: Finding patterns in tens.

Powers of 10:

1. Use your calculator to help you find the products below.
   (a) $10 \times 10 =$ ______________________________
   (b) $10 \times 10 \times 10 =$ ______________________________
   (c) $10 \times 10 \times 10 \times 10 =$ ______________________________
   (d) $10 \times 10 \times 10 \times 10 \times 10 =$ ______________________________
   (e) $10 \times 10 \times 10 \times 10 \times 10 \times 10 =$ ______________________________

2. Without using your calculator, anticipate the product:

   $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 =$ ______________________________

3. Describe a shortcut for multiplying tens.

Place Value:

4. Find the products below:
   (a) $4 \times 100,000 =$ ______________________________
   (b) $7 \times 1000 =$ ______________________________
   (c) $2 \times 100 =$ ______________________________

5. Write out the names of your answers to number (4).
   (a) ______________________________
   (b) ______________________________
   (c) ______________________________

6. What is the name of the number 407,200?

Notice how the names of the numbers in (6) match the numbers you found in (5). The 4 represents $4 \times 100,000$, the 7 represents 7×1000 and the 2 represents 2×100.

2 Place Value

There are 10 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. A number might have more than one digit. The digits in a number have different values associated with them depending on the location of the digit in the number.

For example:

256 has three digits and 5 has a value of fifty or $5 \times 10 = 50$. As you read the number, the place values are spoken as, two-*hundred* fif*ty* six.

24.89 has 4 digits and 8 has value: $8 \times 0.1 = 0.8$

Exercise 2-1: Associating digits with money. *$1 bill, $10 bill, $100 bill, $1000 bill*

Leticia wants to buy a new motorcycle that costs $6,327.25

1. How many digits are in 6,327.25?

2. Which digit has the largest place value?

3. What is the value of the digit with the largest value?

4. Which number has the smallest value and what is its value?

5. Leticia pays cash for the bike and the bills she can get from the bank are in the denominations $10,000, $1000, $100, $10, $1, as well as dimes and pennies. How many of each denomination will she have to pay to cover the exact price if she uses the minimum number of bills?

6. The value of each denomination of the bills Leticia used in question (5) is the place value of each digit within the number 6,327.25. Name the place value of each digit.

As you see in exercise (2-1) there is a monetary value associated with each digit of a number according to the placement of the number. For example, the value of the digit 6 is thousands since you need 6 one thousand dollar bills and the value of the 5 is 5 hundredths because you need 5 pennies and each penny is worth one hundredth of a dollar.

2.1 Introducing exponents with base 10

The following chart explains some of the place values in the number system:

Numbers to show place value can be shown by powers of 10, such as:

$1 \cdot 10^9 = 1 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1,000,000,000$	one billion
$1 \cdot 10^8 = 1 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100,000,000$	one hundred million
$1 \cdot 10^7 = 1 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10,000,000$	ten million
$1 \cdot 10^6 = 1 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1,000,000$	one million
$1 \cdot 10^5 = 1 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100,000$	one hundred thousand
$1 \cdot 10^4 = 1 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10,000$	ten thousand
$1 \cdot 10^3 = 1 \cdot 10 \cdot 10 \cdot 10 = 1,000$	one thousand
$1 \cdot 10^2 = 1 \cdot 10 \cdot 10 = 100$	one hundred
$1 \cdot 10^1 = 1 \cdot 10 = 10$	ten
$1 \cdot 10^0 = 1 \cdot 1 = 1$	one
$1 \div 10 = 0.1$	one tenth
$1 \div 100 = 0.01$	one hundredth
$1 \div 1000 = 0.001$	one thousandth

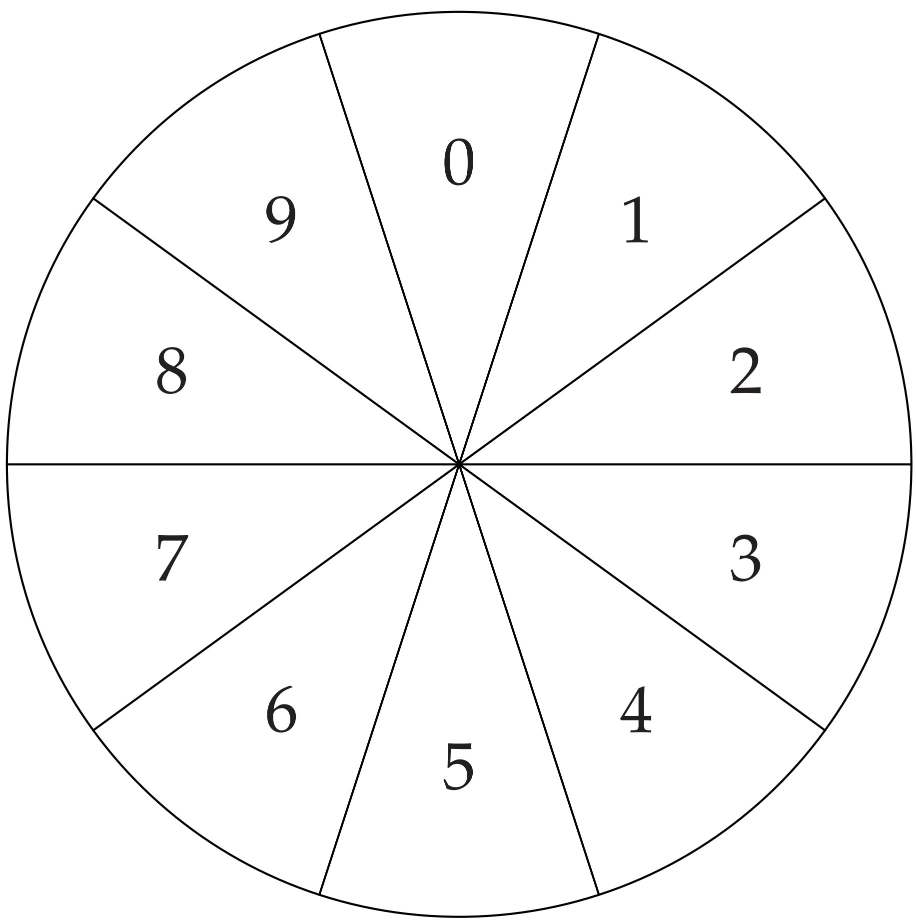
Exercise 2-2:

Find the place value and **value** of each digit underlined in following chart:

number	place value	value
$23\underline{7},669$		
$\underline{9},932,210$		
$1\underline{0}2$		
$86\underline{5},106$		
$65.2\underline{9}3$		
$10.\underline{2}26$		
$0.\underline{2}26$		
$\underline{6}50,900,563,115.07$		
$1127.0\underline{8}335$		

Group Activity:

1. Each group needs a paper clip and 3 different pieces of paper.
2. Open the paper clip from one side and use it as a spinner.
3. Each member draws 3 lines on his or her paper.
4. Each time one of the members spins the paper clip and each member records the number on one of the 3 lines that he or she has drawn.
5. After 3 spins compare your 3-digits numbers.
6. The largest 3-digit number wins the turn.



Exercise 2-3:

Notice that $235 could come in 235 $1 bills, or 2 $100 bills, 3 $10 bills and 5 $1 bills , or 23 $10 bills and 5 $1 bills.

1. Which combination has the fewest number of bills?

2. Find three combinations of bills to make $5,628.

3. Find three combinations of bills to make $63,921.

4. Find three combinations of bills to make $307,995.

5. If you had 21 $1000-bills, no $100-bills, 26 $1-bills, no dimes and 13 pennies and exchanged them for the fewest number of bills/coins that represented the same amount, how much of each kind of bill/coin would you have? How much would your money be worth? In the bank record below, record how much money you have, and how many of each type of bill/coin you would have when using the fewest number of each to represent the amount.

Total value of money	$10,000's	$1,000's	$100's	$10's	$1's	dimes	pennies
________	____	____	____	____	____	____	____

6. When writing money in dollars and cents format, where do you put the decimal place? (between the count of which of the bills/coins?)

7. What would be the largest number you could make with the digits 0, 1, 2, 5, 8, 9 (with no decimal points)?

8. What would be the smallest number you could make with the digits 0, 1, 2, 5, 8, 9 (with no decimal points)?

9. What would be the smallest 6-digit number with 2 digits after the decimal point you could make with the digits 0, 1, 2, 5, 8, 9?

10. What would be the largest 6-digit number with 2 digits after the decimal point you could make with the digits 0, 1, 2, 5, 8, 9?

11. If you had the digits 0, 1, 2, 5, 8, 9 to make a 6-digit number, what would be the smallest number you could make with the 6 digits? You choose the number of digits after the decimal place.

12. In the number $3,461.85, which digit has the highest place value (i.e., accounts for the most money?)

13. In the number $0.08, which digit has the highest place value?

14. In the number $4501.32, what is the place value of the digit 0?

15. In the number $673.47, what is the place value of the digit 4?

16. In the number $86, 793.64, what digit is in the ten-thousands place?

17. In the number $86, 793.64, what digit is in the hundreds place?

18. In the number $86, 793.64, what digit is in the tens place?

19. In the number $86, 793.64, what is the place value of the digit 9?

20. What is the largest 5-digit number?

21. What is the smallest 5-digit number (without the decimal point)?

22. What is the largest 5-digit number that you can make with the digits 6, 6, 1, 0, and 8 each used exactly once?

23. What is the smallest 5-digit number that you can make with the digits 6, 6, 1, 0, and 8, each used exactly once? You may include a decimal point.

2.2 Saying and Writing Numbers in Words

To read and write whole numbers: The whole numbers are the numbers in the sequence 0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, continuing indefinitely. When we write whole numbers in standard form we separate a group of three digits by a comma. Each group of three digits forms what is called a **period**. Each period has a particular name. We say the name of a period after we read all the numbers in that period, except the last period before the decimal point.

Example 1: Read 8,412,769,215.

Solution: Eight *billion*, four hundred twelve *million*, seven hundred sixty nine *thousand*, two hundred fifteen. (The name of each period is in *italics*. Notice that the last period doesn't have a name.)

To read and write decimal numbers: We read the whole number part exactly the same way as before, then read the decimal as "and" followed by the name of the number to the right of the decimal point using the place value of the last digit as the number's value.

Example 2: Read 65,389,237,542.029.

Solution: Sixty five billion, three hundred eighty nine million, two hundred thirty seven thousand, five hundred forty two, and twenty nine thousandths. (Notice that the name of the decimal part is *thousandths* since the last digit, the 9, is in the thousandths place.)

Exercise 2-4: Place commas to separate the periods, then write out the number in words:

1. 2863
2. 76.218
3. 28990.02
4. 3400651899
5. 23678.9352
6. 1000267
7. 9664822.2
8. 1273579056333

3 Rounding and Significant Digits

When is rounding a good idea, and when is it not?

Examples:

1. The U.S. trade deficit in June 2005 was $58.8 billion.

2. Measurement of the distance you drive to LA.

3. Measurement for rope to tie down a load.

4. Measurement for wood in making a picture frame.

5. The dose of Benadryl for a toddler is 0.5mg/pound (12.5 mg = 1 tsp)

6. The dose of the arthritis drug prescribed to each patient is 0.022 cc per day.

7. Jenisa calculated her share of the lunch that she had with her high school friends yesterday and the figure was $12.347.

8. Computing your grade.

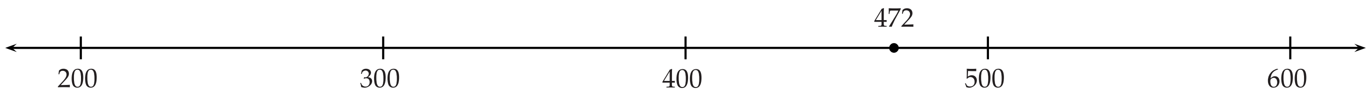
9. Heart medication at 5 mg.

10. A cup of flour to make a cake.

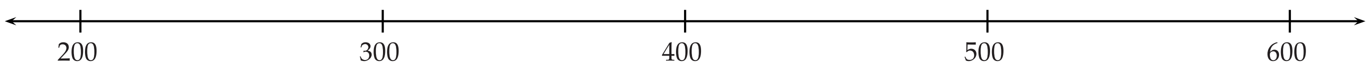
There are two basic types of rounding: rounding to a particular place value, and rounding to a particular number of significant digits. The first is when you know what level of acuracy you want, for example rounding to the nearest cent or to the nearest whole dollar. Rounding to a number of significant digits is usually used to simplify calculations by lowering the number of non-zero digits left in each number to the number desired.

Graphing and Rounding Activity

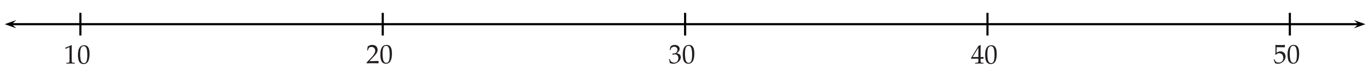
Graph the following numbers on the number line, for example, to graph 472, place a dot where 472 should go, then write 472 next to the dot as in the figure:



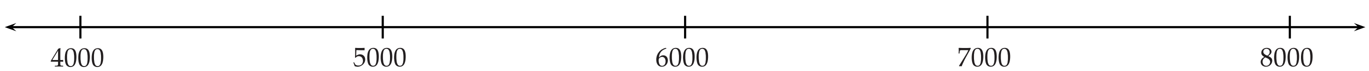
1. Graph 220, 517, 350, 490:



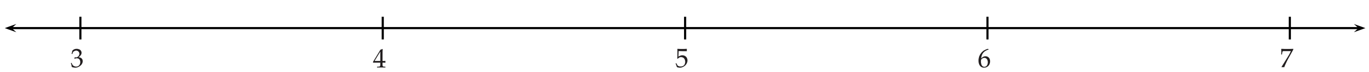
2. Graph 28, 17, 35, 46:



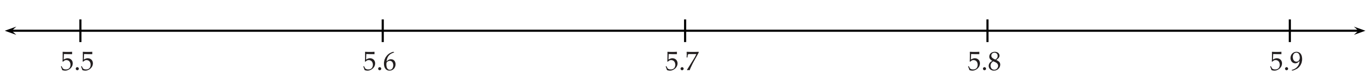
3. Graph 5593, 7200, 6500, 4207:



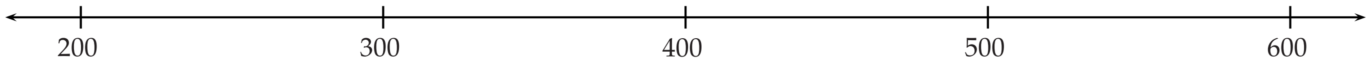
4. Graph 3.5, 5.2, 6.6, 4.1:



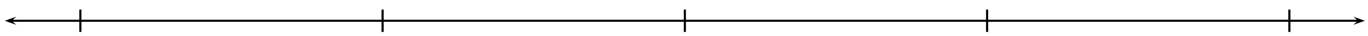
5. Graph 5.72, 5.63, 5.88, 5.55:



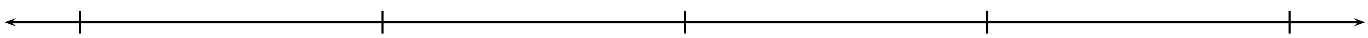
Mark each number line by counting by the indicated amount. For example, to mark the number line by counting by hundreds starting at 200, you would mark the number line like the following:



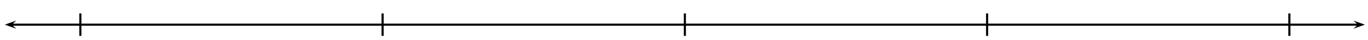
1. Count by hundreds, starting at 500:



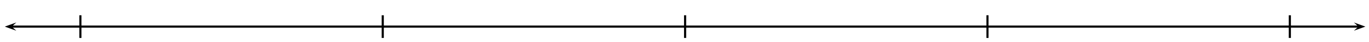
2. Count by tens, starting at 30:



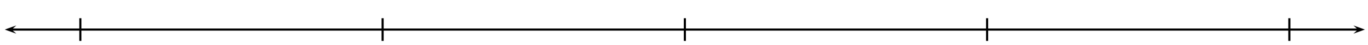
3. Count by ones, starting at 8:



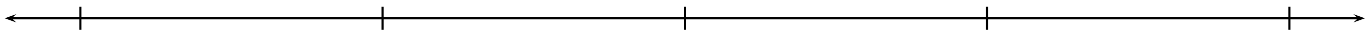
4. Count by tens, starting at 420:



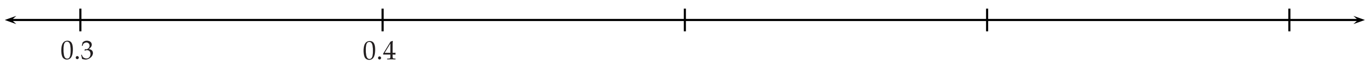
5. Count by hundreds, starting at 2100:



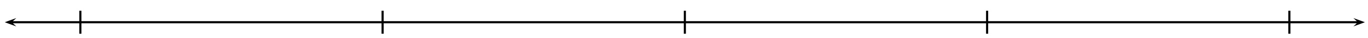
6. Count by tens, starting at 2100:



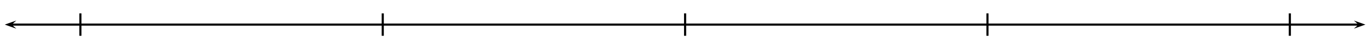
7. Count by tenths (0.1’s), starting at 0.3:



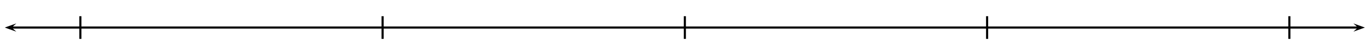
8. Count by tenths, starting at 1.8:



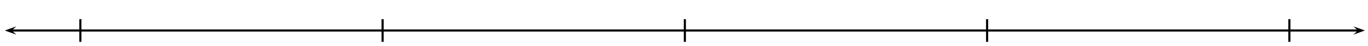
9. Count by tenths, starting at 22.5:



10. Count by hundreds, starting at 34,800:



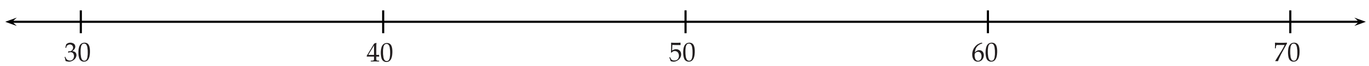
11. Count by thousands, starting at 998,000:



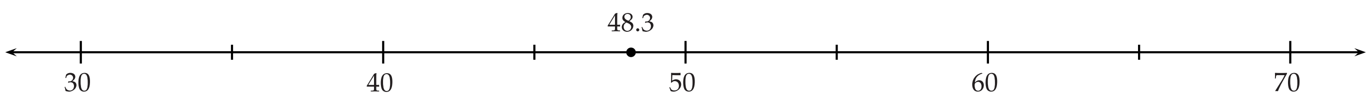
To round numbers to a particular decimal place, follow the following steps:

1. Determine the correct decimal place (tenths, ones, tens, hundreds, etc.)
2. Draw a number line, and count by the determined decimal place starting at a number close to but not more than the number you are rounding.
3. Graph the number that you are rounding on the number line.
4. The rounded value is the count number that is closest to your graphed number. If the graphed number is exactly in the middle of two count numbers, round to the higher number.

For example: To round 48.3 to the nearest ten, count by tens starting at 30 like the following:

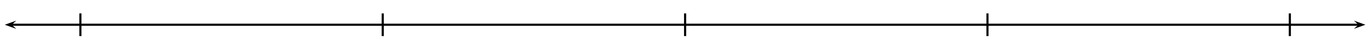


Then, graph 48.3, and determine which number is closest...it helps to put a tick mark at the half-way points:



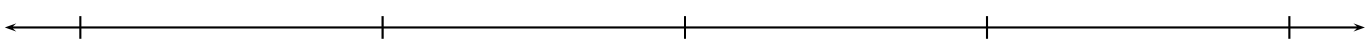
48.3 is to the right of the half-way point between 40 and 50, so it's closer to 50. Therefore, 48.3 rounded to the nearest ten is 50.

1. Round 286 to the nearest hundred:



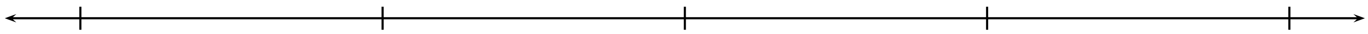
Final answer:________________

2. Round 5.37 to the nearest one:



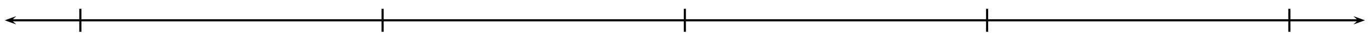
Final answer:________________

3. Round 5.37 to the nearest tenth:



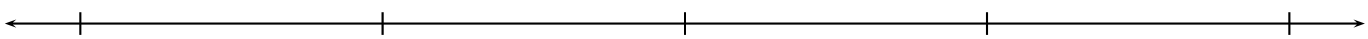
Final answer:________________

4. Round 5.37 to the nearest ten:



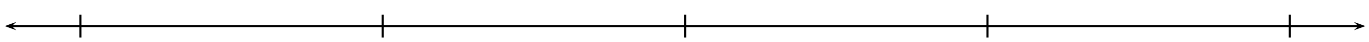
Final answer:________________

5. Round 5.37 to the nearest hundred:



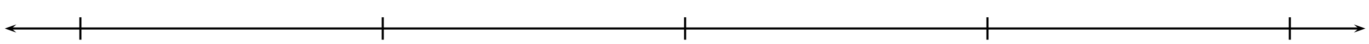
Final answer:________________

6. Round 2358 to the nearest hundred:



Final answer:________________

7. Round 2358 to the nearest thousand:



Final answer:________________

3.1 Significant Digits

Significant digits are used to help us indicate accuracy in numbers. When we use a measuring tape to measure a block of wood, the measurements are only as accurate as the tape allows us to make them. Most measuring tapes have only one or two decimal places of accuracy marked on them since we are unable to see in finer detail. People have designed other tools to use when they need more accuracy.

When we say "round a number to one significant digit" or to two significant digits, and so on, we determine one, two, or however many significant digits in the following manner:

Reading from left to right, the first nonzero number and all the digits following it are significant.

Example: The number 340.69 has five significant digits. The first significant digit is 3. The first two significant digits are 3 and 4.

Example: The number 0.034069 also has five significant digits. The zeros on the left are not considered significant for the purposes of rounding.

Example: In each of the following numbers, the digits that are in bold and their significant digits are being indicated or the place value of the digit is indicated.

- **3**45.69 1 significant digit is in bold.
- **45**72.28 2 significant digits are in bold.
- **34.59** 4 significant digits are in bold.
- 0.0**275** 4 significant digits are in bold.
- 23.**7**9 The tenth's place is in bold.

What is the appropriate significant digit to round to in different numbers?

That depends on what kind of accuracy is needed in a particular situation.

- When measuring the radius of the earth you need less accuracy than measuring the dimensions of a room that you want to carpet.
- When you have to perform a series of calculations in a problem it is a good idea to round off at the end to get a more accurate solution, as opposed to estimating a solution, when you need to round off before calculations. When you are rounding before to estimate, you are making your problem easier to calculate, but be aware that you are sacrificing accuracy.
- Your final answer is never more accurate than the information you were given.

The idea of rounding is to choose a level of acuracy, either a number of significant digits, or a particular decimal place. Your rounded number is the closest number to your original that has the level of accuracy desired. The following process can be used to round:

Process of rounding if the digit you are rounding to is to the left of the decimal point:

1. Mark the digit that you are going to round to.
2. Look at the digit to the right of the digit that you marked in step 1.
3. If the digit in step 2 is a 5, 6, 7, 8 or 9, add one to the digit that you marked in step 1 and make all the other digits after that zeros.
4. If the number in step 2 is 0, 1, 2, 3 or 4, just make all digits after the digit you marked in step 1 zeros. Any digits to the right of the decimal point and the decimal point itself are not written.

Exampe 1:

Round 74392 to the nearest hundred.

Solution:

The digit marked for step one is the 3 in the hundreds place: 74**3**92 .
The digit to look at for step 2 is the 9: 743**9**2 .
Since 9 is 5 or more, we add 1 to the 3 to get a 4 in the hundreds place. The digits to the right are changed to zeros. 74392 rounded to the nearest hundred is: 74400.

Exampe 2:

Round 276, 395, 661 to three significant digits.

Solution:

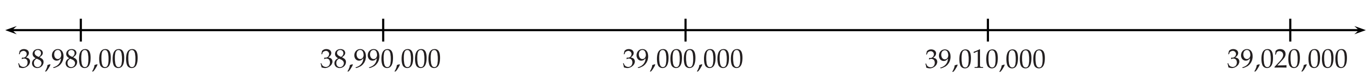
The third significant digit is in the millions place, so the digit marked for step 1 is the 6 in the millions place: 27**6**,395,661 .
The digit to look at for step 2 is the 3: 276,**3**95,661 .
Since the digit marked in step 2 is 3 which is less than 5, we just change 3, 9, 5, 6, 6 and 1 to zeros. 276,395,661 rounded to the nearest million is 276,000,000 which is often written 276 million.

Exampe 3:

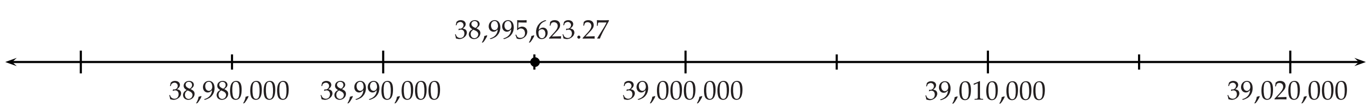
Round 38,995,623.27 to the nearest ten-thousand.

Solution:

The digit marked for step 1 is the 9 in the ten-thousands place: 38,9**9**5,623.27 .
The digit to look at for step 2 is the 5: 38,99**5**,623.27 .
Since the 5 in step 2 is 5 or more, we add 1 to the 9. But this is tricky! A better way of looking at it is that we are adding 1 to the rest of the number, that is, adding 1 to 3899 to get 3900. Write the rest of the digits as zeros, leaving out the decimal point and the digits to the right of the decimal point. This is best seen for many by looking at the number line. If we count by 10,000's starting just below our original number, we get the following:



Our number is right in between 38,990,000 and 39,000,000. Notice, it is a little closer to the 39 million. By our proceedure, the digit 5 indicates that we round up:



Process of rounding if the digit you are rounding to is to the right of the decimal point:

1. Mark the digit that you are going to round to.
2. Look at the digit to the right of the digit that you marked in step 1.
3. If the digit in step 2 is a 5, 6, 7, 8 or 9, add one to the digit that you marked in step 1 and drop all the other digits after that.
4. If the number in step 2 is 0, 1, 2, 3 or 4, just drop all digits after the digit you marked in step 1.

Exampe 1:

Round 325.7496 to the nearest hundredth.

Solution:

The digit marked for step one is the 4 in the hundredths place: 325.7**4**96 .
The digit to look at for step 2 is the 9: 7325.74**9**6 .
Since 9 is 5 or more, we add 1 to the 4 to get a 5 in the hundredths place. The digits to the right are dropped. 325.7496 rounded to the nearest hundredth is: 325.75.

Exampe 2:

Round 12.68531 to five significant digits.

Solution:

The fifth significant digit is in the thousandths place, so the digit marked for step 1 is the 5 in the thousandths place: 12.68**5**31 .
The digit to look at for step 2 is the 3: 12.685**3**1 .
Since the digit marked in step 2 is 3 which is less than 5, we just drop the 3 and the 1. 12.68531 rounded to five significant digits is 12.685.

Exampe 3:

Round 439.9907 to four significant digits.

Solution:

The fourth significant digit is in the tenths place, so the digit marked for step 1 is the 9 in the tenths place: 439.**9**907 .
The digit to look at for step 2 is the 9 in the hundredths place: 439.9**9**07 .
Since the digit marked in step 2 is 9 which is 5 or more, we round up to 440.0. We need the extra 0 in the tenth's place to indicate how accurately we have rounded. Rounding to five significant digits would have required an extra zero, and the answer would have been 440.00.

Exampe 4:

Round 1.38562 to three decimal places.

Solution:

The third decimal place is the thousandths place, so the digit marked for step 1 is the 5: 1.38**5**62 .
The digit to look at for step 2 is the 6 in the ten-thousandths place: 1.385**6**2 .
Since the digit marked in step 2 is 6 which is greater than or equal to ($\geq$) 5, we round up to 1.386.

Exercise 3-1: Fill in the table to practice rounding:

The Number to Round off and the Level of Rounding	Underline the Digit(s)	The Next Digit	Round **Up** or **Down**	Answer
1) 207.845 Round to 3 significant digits	$\underline{207}.845$	8	Up, since $8 \geq 5$	208
2) 0.0278273 Round to 3 significant digits	$0.0\underline{278}273$	2	Down, since $2 < 5$	0.0278
3) 18.0974 Round to 1 decimal place	$18.\underline{0}974$	9	Up, since $9 \geq 5$	18.1
4) 18.0974 Round to 1 significant digit	$\underline{1}8.0974$	8	Up, since $8 \geq 5$	20
5) 23.0328 Round to the nearest 100th	$23.0\underline{3}28$	2	Down, since $2 < 5$	23.03
6) 5671.983 Round to the nearest 100				
7) 5671.983 Round to the nearest 100th				
8) 1098.0067 Round to 3 significant digits				
9) 36.0409 Round to the nearest 100th				
10) 36.0409 Round to the nearest 1000th				
11) 99.9986 Round to 1 significant digit				
12) 99.9986 Round to the nearest 10th				
13) 0.00009826 Round to 1 significant digit				
14) 0.00009826 Round to 2 significant digits				
15) 0.00009826 Round to the nearest 100th				
16) 0.00009826 Round to the nearest 1000th				
17) 19.87 Round to the nearest 10				
18) 19.87 Round to 1 significant digit				
19) 6.854 Round to 2 decimal places				
20) 6.854 Round to 2 significant digits				
21) 345.65 Round to 1 significant digit				
22) 90,099.99 Round to the nearest 100				
23) 90,099.99 Round to the nearest 10th				
24) 3.0865 Round to the nearest one				
25) 3.0865 Round to the nearest 10				
26) 6,970,493.85 Round to the nearest 10,000				

4 Addition of Whole Numbers, Decimals, and Fractions

4.1 Money Activity:

Assume the bank has piles of bills only in the denominations $10,000, $1000, $100, $10, $1, dimes and pennies. Each of the three members in your group take bills to total the following:

- Member #1: $25,693.12
- Member #2: $9,275.35
- Member #3: $10,968.78

When you take your money, make sure that the number of bills in each denomination is less than ten. For each of the following problems, start each member with their original amount of money and with their original number of each type of bill.

1. (a) How much of each bill do member #1 and member #2 have together? Record the number of bills in each denomination.

$10,000	$1000	$100	$10	$1	dimes	pennies

(b) Keeping the total amount of money the same exchange each denomination if necessary to keep the number of bills in each denomination less than ten. Record the new number of bills in each denomination.

$10,000	$1000	$100	$10	$1	dimes	pennies

(c) How much money do they have together?

2. (a) How much of each bill do member #1 and member #3 have together? Record the number of bills in each denomination.

$10,000	$1000	$100	$10	$1	dimes	pennies

(b) Keeping the total amount of money the same exchange each denomination if necessary to keep the number of bills in each denomination less than ten. Record the new number of bills in each denomination.

$10,000	$1000	$100	$10	$1	dimes	pennies

(c) How much money do they have together?

3. (a) How much of each bill do member #2 and member #3 have together? Record the number of bills in each denomination.

$10,000	$1000	$100	$10	$1	dimes	pennies

   (b) Keeping the total amount of money the same exchange each denomination if necessary to keep the number of bills in each denomination less than ten. Record the new number of bills in each denomination.

$10,000	$1000	$100	$10	$1	dimes	pennies

   (c) How much money do they have together?

4. (a) How much of each bill do all three members have together? Record the number of bills in each denomination.

$10,000	$1000	$100	$10	$1	dimes	pennies

   (b) Keeping the total amount of money the same exchange each denomination if necessary to keep the number of bills in each denomination less than ten. Record the new number of bills in each denomination.

$10,000	$1000	$100	$10	$1	dimes	pennies

   (c) How much money do they have together?

5. Explain how you found the combined monies (sums) above.

6. Compare your answers with another group in your class. Compare the number of each denomination from each question before and after combining monies. Are there any common ways of totaling that your groups followed?

7. What have you observed about the process of addition?

4.2 The Addition Process

During the preceeding activity, you observed that when we add money, we combine the bills with the same denominations together and every time that the number of bills in each denomination reaches 10, we replace that with one bill from the next higher denomination, that is, the one to the left of it in the place value table. For example, ten dimes is replaced by 1 dollar or ten $10 bills is replaced by one $100 bill.

Vocabulary:

- The operation is called **addition**.
- Each of the numbers that we add together is called an **addend**.
- When two numbers are added together, the answer is called the **sum** or **total**.

Adding whole numbers or decimals together:

1. Line up each addend so that the digits with the same place values are in the same column.
2. Start adding from the rightmost column.
3. If the sum of the numbers in a column is 9 or less, record the sum and proceed to the next column.
4. If the sum of the digits is 10 or more, write the rightmost digit of the sum under the column you are adding and add the other digits to the appropriate columns to the left of it. This amount is "carried over" to the next higher place value column.
5. Repeat until all the digits of all the columns are added.

Example 1: Add the following two numbers:

$25,693.12 + 9,275.35$

Solution:

Place value	10,000	1000	100	10	1	decimal point	$\frac{1}{10}$	$\frac{1}{100}$
Carry over	1		1					
First number	2	5	6	9	3	.	1	2
Second number		9	2	7	5	.	3	5
Sum:	3	4	9	6	8	.	4	7

So that the answer is 34,968.47

Example 2: Add the following three numbers:

$76,926.8 + 685.87 + 93,768.79$

Solution:

Place value	100,000	10,000	1000	100	10	1	decimal point	$\frac{1}{10}$	$\frac{1}{100}$
Carry over	1	1	2	1	2	2		1	
First number		7	6	9	2	6	.	8	
Second number				6	8	5	.	8	7
Third number		9	3	7	6	8	.	7	9
Sum:	1	7	1	3	8	1	.	4	6

So that the answer is 171,381.46

4.3 Addition Activities:

Pig: Each group chooses a number between 30 and 55. Each player may roll the die as many times as she likes. Player adds the face value of die to the sum so far and tries to get as close to the chosen target number as possible. Here's the catch: if a player rolls same number two times in a row, they lose their turn and their points! (For a variation: the player keeps their points so far and just loses their turn). E.g., a roll might look like: $3+5+6+2+5+4+4$. The player loses their points and their turn because they rolled two fours in a row before stopping or reaching the target. It is possible to win this game with a roll of 1 — if the other players go out first.

Exercise 4-1: Value of Words:

By assigning a value to each of the letters of the alphabet, all of a sudden, words have value aside from their ability to help us communicate. For the problems below, use the values given in the chart for each letter. The value of a word is the total of the values of its letters.

A=$1	G=$7	L=$12	Q=$17	V=$22
B=$2	H=$8	M=$13	R=$18	W=$23
C=$3	I=$9	N=$14	S=$19	X=$24
D=$4	J=$10	O=$15	T=$20	Y=$25
E=$5	K=$11	P=$16	U=$21	Z=$26
F=$6				

1. Show that "Skyline" is worth $95.

2. How much is "Pacifica" worth?

3. Who has the most expensive name (last or first; not both)?

4. Find a 3-letter word that has:
   (a) the cheapest value
   (b) the most expensive value

5. What is the most expensive word that you can find?

Exercise 4-2: Complete the following addition table:

+	0	1	2	3	4	5	6	7	8	9	10
0											
1											
2											
3											
4											
5											
6											
7											
8											
9											
10											

1. Find at least four different patterns in this table.

2. Notice that there are two ways to get 2 as a sum: 0 + 2 (we'll assume 2 + 0 is the same) and 1 + 1.

   (a) How many different ways are there to get the number 3 as a sum? ________________

   (b) How many different ways are there to get the number 4 as a sum? ________________

   (c) How many different ways are there to get the number 5 as a sum? ________________

   (d) How many different ways are there to get the number 6 as a sum? ________________

   (e) How many different ways are there to get the number 7 as a sum? ________________

   (f) How many different ways are there to get the number 8 as a sum? ________________

   (g) How many different ways are there to get the number 9 as a sum? ________________

3. What patterns do you notice in your answers to question 2? (List at least three). Is there an easy way to use the table to find the different sums? Explain.

4. How many different ways are there to get the number 10 as a sum? ________________

5. From your observations in the previous questions you should be able to guess the number of different ways to write 100 as a sum:

Exercise 4-3: Add the following numbers:

1. $23,409.43 + 12,000.99$
2. $678,222.099 + 7,392.3$
3. $731 + 2.006$
4. $987,962.051 + 16 + 687.084$
5. $11,276 + 0.038 + 2.38$
6. $99 + 10,268.5 + 28.03$
7. $0.00397 + 2.015 + 69.11$
8. $763.19 + 2,763.019 + 3.0019$
9. $287,309,227.56 + 199.99$
10. $0.48 + 268 + 60.46$

4.4 Estimation or Approximation

The Process of Estimating with Addition:

- Round each number in the estimation problem to one significant digit.
- Add the numbers from step 1.

Example 1: Estimate the sum:

$5,629 + 236$

Solution:

5,629 rounded to one significant digit is 6,000

236 rounded to one significant digit is 200

The sum of 6000 and 200 is 6200

The estimate of the sum of 5,629 and 236 is 6200

Example 2: Estimate the sum:

$34,296.5 + 28$

Solution:

34,296.5 rounded to one significant digit is 30,000

28 rounded to one significant digit is 30

The sum of 30,000 and 30 is 30,030

The estimate of the sum of 34,296.5 and 28 is 30,030

Exercise 4-4: Approximate Sum Practice

For each of the following sums:

- Estimate the sum by rounding each number to one significant digit before adding.
- Find the actual sum.
- State whether the actual sum is $>$, $=$, or $<$ the estimate.

1. $259.05 + 12.99$

2. $3,759.12 + 4,250.7 + 66$

3. $27,369 + 8.64 + 150$

4. $24,269.5 + 35,253.05 + 41,299.8$

5. $127,280 + 196 + 25$

6. $5.099 + 0.89 + 2.01$

7. $7,831,004.05 + 32,890.9$

8. $2,224,766 + 5,689,20 + 1,920,761$

9. $2,256,703 + 156.56$

10. $276,107 + 24,109.5 + 3,655.7 + 390.7$

Magic Squares

Look at the square grid of numbers shown below. See if you can find anything unusal about the numbers in the square.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

This square is called a magic square because the numbers in each row, in each column, and in each diagonal all sum to the same thing, called the magic number. In this case the magic number is 34. The artist Albrecht Dürer (1471 - 1528) is credited with being the first European to publish a magic square, by including the square shown above in his copper engraving, "Melancholia I". See if you can find the year the piece was completed, hidden somewhere in the magic square.

1. The magic square below is composed of the digits 1 through 9 (exactly once each) but some have been left out. Fill in the missing numbers in the correct places.

8		6
	5	
4		2

2. The numbers 1, 3, 7, 8, 9, 10, 14, and 15 have been removed from the magic square below. Put them back in the correct locations.

	14		4
11		6	
	10		5
13	2		

3. The numbers 6, 10, 11, 12, 13, 14, 15, and 16 have been removed from the magic square below. Put them back in the correct locations.

9		3	
4			5
	1	8	
7			2

4.5 Addition in Real Life

In a math class, we see the word "sum" used to indicate that we are supposed to add. In real life, the operation of addition can be implied in many other ways. Some key words to look for are total, increase, or perimeter, but sometimes none of these are used, and you have to reason, from the context, that addition is implied.

Example 1: Marco and Mary went Christmas shopping. They found an alarm clock for $24.99, a CD for $14.45, a pillow for $32.20, a rug for $18.69, and a game for $11.50. Estimate the pre-tax cost for all of these items by rounding each value to the nearest whole dollar before computing, then find the actual pre-tax cost.

Solution:

It's not stated explicitly in the problem, but we can reason that the cost for all of the items is the sum of each cost. We begin by rounding each to the nearest whole dollar:

$\$24.99 \approx \$25$

$\$14.45 \approx \$14$

$\$32.20 \approx \$32$

$\$18.69 \approx \$19$

$\$11.50 \approx \$12$

Then, we add the rounded values, $25 + 14 + 32 + 19 + 12 = 102$, to get our estimate of $\$102$.

Finally, we add up the numbers with the cents included: $24.99 + 14.45 + 32.20 + 18.69 + 11.50 = 101.83$, which is close to our estimate, so we are confident that we didn't make a careless error! The pre-tax cost of all of the items is: $101.83.

Example 2: Sammi was excited when the value of her stock went up. At the beginning of the month, the value was $42.90 per share, but when she checked at the end of the month, the website indicated that the value had increased by $2.73 per share. What was the per share value of her stock at the end of the month?

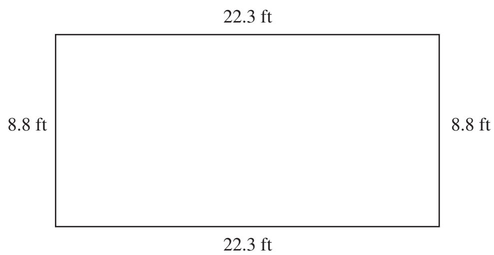
Solution:

To find a new value of something after an increase, and the amount of the increase to the starting value. First, even though we were not told to estimate, it's a good idea, so we round each number to the nearest dollar before adding: $43 + 3 = 46$ for an estimate of $\$46$. When we calculate the actual sum, we get:$42.90 + 2.73 = 45.63$. The new value of her stock is: $45.63 per share. Notice we have included the units in the answer instead of just writing 45.63.

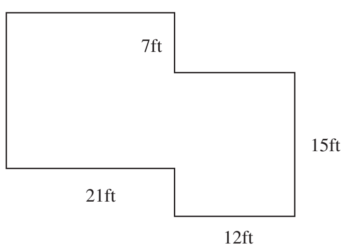
Example 3: The Community Gardening Society wanted to make a short fence to surround their garden. The plot is a rectangle that measures 22.3 feet by 8.8 feet. How many feet of fencing do they need to buy?

Solution:

The amount of fencing that they need is the total distance around the rectangle, or, what is called the *perimeter* of the rectangle. Since it's a rectangle, the side measurements are repeated on the opposite sides (see figure), so the total is $22.3 + 8.8 + 22.3 + 8.8$. A quick estimate gives us $20 + 9 + 20 + 9 = 58$ feet, and with a little more work, the actual perimeter is 62.2 feet.

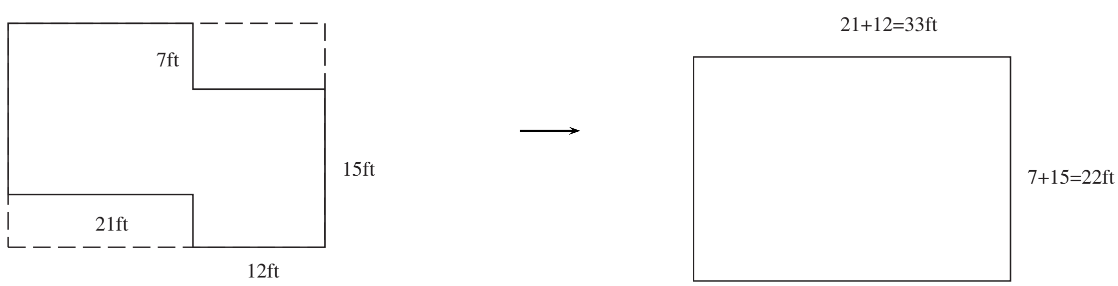


Example 4: Find the perimeter of the following figure:



Solution:

The Perimeter of more complicated figures can be computed with a little effort, and with what we math folks call *problem solving*. The first thing to realize, is that the perimeter of this figure is the same as the perimeter of the rectangle that it fits into. (Talk with your classmates or your instructor until you understand why this is true!)



From this picture we see that the dimensions of the surrounding rectangle are 33 feet by 22 feet, giving us our estimate of $30 + 20 + 30 + 20 = 100$ feet, and our actual value of $33 + 22 + 33 + 22 = 110$ feet.

Exercise 4-5: Real Life Addition Exercises

For the following exercises, write the sum that is needed to answer the question, then estimate the sum by rounding each addend to one significant digit, then find the actual sum. Write units on the answer if appropriate.

1. Clarence goes to the store and starts putting items in his cart. As he is approaching the checkout line, he wants to know what his total charge will be. The items in his cart are:

   sugar \$1.75, eggs \$2.50, bread \$2.25, milk \$3.19, cheese \$4.49, cereal \$3.69,
   2 boxes of Mac and Cheez at \$0.79 each, soup \$2.45, juice \$5.50, gum \$0.44.

   What will be the total?

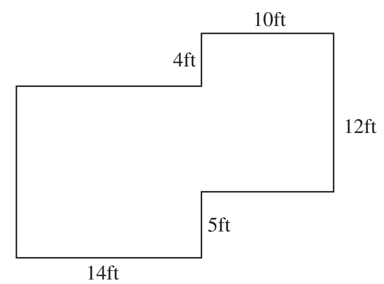
2. Hector wants to build a frame for a picture that is 8 inches tall and 10 inches wide. He plans to buy a single piece of wood and cut it up to make the frame. How long should the piece of wood be?

3. As a sales rep, Marta has to drive a lot. One day, she drove 11.8 miles from San Francisco to Oakland, then she went 41.6 miles from Oakland to Napa. She had lunch in Napa, then drove 39.3 miles to San Rafael, and ended her day in San Jose, which is 36.8 miles from San Rafael. How far did she drive that day?

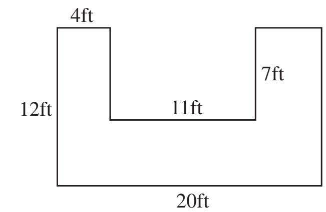
4. After finally convincing his spiky-haired boss, Dilbert was able to get a raise of \$1945 per year from his old salary of \$40,275 per year. What is his new salary?

5. Find the perimeter of each figure.

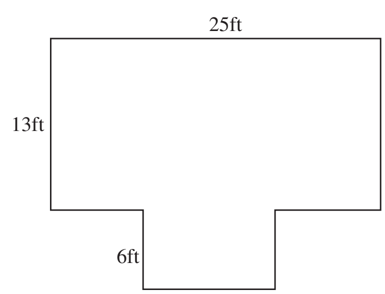
(a)



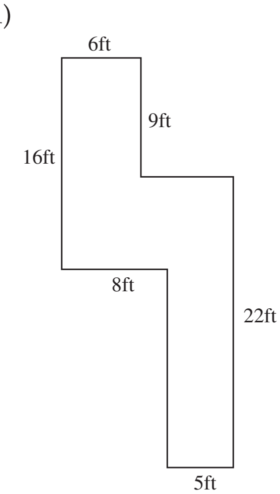
(b)



(c)

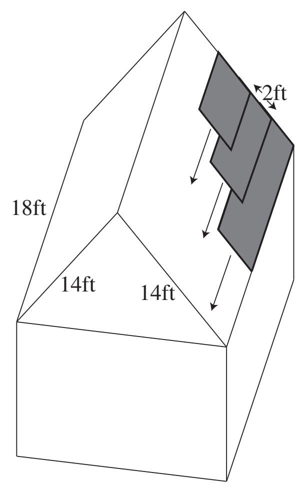


(d)



6. One month, Henry and Herietta's bills were out of control. Their PG&E bill was $186.82, their phone bill was $62.37, their rent was the normal $1245, their credit card minimum payment was $89.79, and their Chevron charge was $183.28. What is the total of these bills?

7. To roof the house below, you must roll out tar paper so that each row overlaps the one below it by 6 inches. If a roll of tar paper is 30 inches tall and 100 feet long, how many rolls will you need to cover this roof?



4.6 Real Quarters Activity

We know from the way it's said, that quarters are a quarter of a dollar. But where is this term coming from? A little history of money in the U.S. will help. The following is an excerpt from http://www.collectsource.com/americas.htm:

> ”The Spanish Dollar quickly became the most popular coin in North America. It is even thought by some that if Washington did throw a coin across the Potomac, it was likely to have been America's First Silver Dollar. Every day commerce was lubricated by this remarkable coin, and the terminology which developed by using it became so deeply embedded in American culture that it remains with us to this day.
>
> **Pieces of Eight**
> Because America's First Silver Dollar was often cut into eight pie shaped "bits" in order to make change, the intact coin became known as a "Piece of Eight." Since the entire Piece of Eight had a value of 8 Reales, each bit was valued at one eighth of the total. Two bits equaled a quarter, four bits a half dollar and six bits three quarters of a dollar. Did you ever spend two bits?—Then you were living the legacy of America's First Silver Dollar! To put the value of America's First Silver Dollar into perspective, an average worker during the colonial era earned about 2 bits a week!"

The following pictures are of coins of this time.

The following activity simulates history in order to introduce fractions and adding fractions.

Materials: 8 breakable silver dollars

Set up: Choose one person in the group to be the merchant, the others to be customers. Customer one starts with 4 dollars, customer two starts with 3 dollars, and customer three (if there are that many in your group) starts with 1 dollar.

Procedure: Act out the following scenario, using the coins as props. Answer each question, using the coins and your discussion to help you. The first customer comes in to buy one shovel for a quarter, 2 pounds of salted meat which costs one quarter per pound, and a blanket which costs 3 quarters.

1. How much is the total charge for customer one?

   Have customer one pay the merchant for the goods.

2. How can the merchant give change?

3. After making the purchase, how many full coins does customer one have left? How many "quarters"?

   Now, customer two buys a live pig for $1 and 3 quarters, and customer three buys a bale of hay for one quarter. After the last two customers pay for their goods, have the merchant give them change in the same way as for customer number one.

4. How many full coins does customer number two have left? How many "quarters"?

5. How many full coins does customer number three have left? How many "quarters"?

6. If the three customers combine their change into one pile, how many full coins do they have altogether? How many quarters?

7. How much money do they have altogether?

8. Write an addition problem, using decimal notation that models the total that you just computed.

9. Now, re-write the sum using fraction notation, that is write $\frac{1}{4}$ for one quarter, $\frac{2}{4}$ for two quarters, etc.

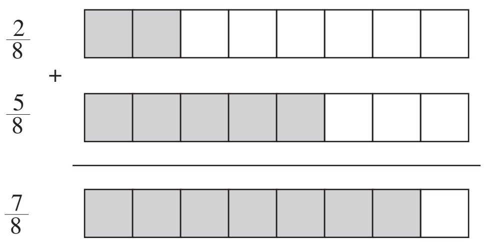
10. Is there more than one way to write the total in fraction form? If so, write the total in as many correct ways as you can think of (up to five).

Fractions

A fraction represents a part of a whole, like a dime is a part of a dollar or a piece of pie is part of the whole pie. The bottom number in a fraction, the *denominator*, tells us the number of pieces the whole was divided into, so the 8 in $\frac{3}{8}$ of a pizza means the whole pizza was cut into 8 pieces. It gives some sense of the size of the pieces, the larger the number, the more pieces, so the smaller each individual piece must be. The top number, the *numerator*, describes the number of pieces we have out of the total. The 3 in $\frac{3}{8}$ of a pizza means we have three slices.

Adding Fractions

As long as the pieces are the same size, adding fractions is no different than adding any other items that are the same. Two dogs plus five more dogs adds up to seven dogs just like two eighths plus five more eighths adds up to seven eighths. In pictures this looks like:



Exercise 4-6: Draw appropriate boxes to demostrate the given fraction addition, then add the fractions. Be sure to make each whole strip the same size, as well as making each piece within the whole the same size. You may need more than one whole for some of the answers.

1. $\frac{1}{3} + \frac{1}{3}$

2. $\frac{2}{5} + \frac{3}{5}$

3. $\frac{3}{10} + \frac{5}{10}$

4. $\frac{1}{4} + \frac{3}{4}$

5. $\frac{3}{8} + \frac{7}{8} + \frac{5}{8}$

6. $\frac{3}{4} + \frac{5}{4}$

7. $\frac{1}{2} + \frac{3}{2}$

8. $\frac{1}{5} + \frac{2}{5} + \frac{4}{5}$

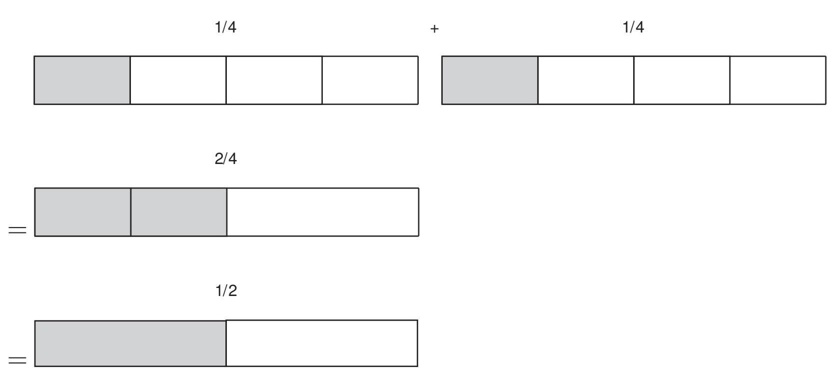
9. $\frac{1}{10} + \frac{7}{10}$

Some of the answers in the last exercise could be written as an *equivalent fraction* with a smaller denominator. Fractions are called *equivalent* when they represent the same amount. For each of the nine answers in the last exercise, check to see if there is an equivalent fraction with a smaller denominator. If so, check "yes", and write the simpler, equivalent fraction. If not, check "no".

Problem	Yes	No	Equivalent Fraction
1			
2			
3			
4			
5			
6			
7			
8			
9			

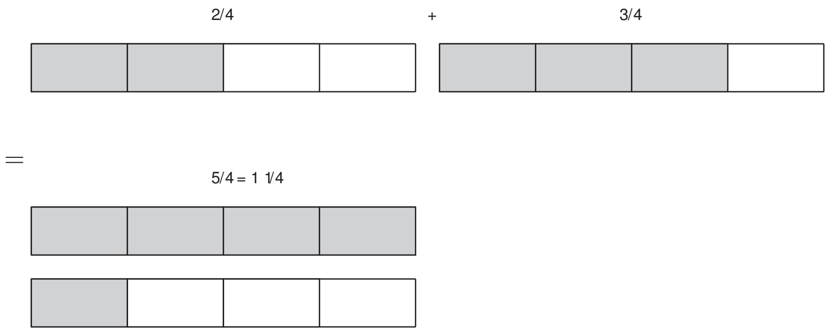
4.7 Equivalent Fractions

When you add money, sometimes you can exchange many smaller bills with a larger bill if you have enough of the smaller bill. The same is true for coins, the fractions that the coins represent, and fractions in general. For example, if you add 1 quarter to 1 quarter you will get two quarters. You may exchange your quarters for one half dollar.



$\frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

Similarly, if you add 2 quarters to 3 quarters you will get 5 quarters. You may exchange four of your quarters for one $1 bill, giving you one and one quarter dollars.



$\frac{2}{4} + \frac{3}{4} = \frac{5}{4} = 1\frac{1}{4}$

Equivalent fractions represent equal amounts.

For example, suppose you wanted to withdraw $\frac{1}{2}$ of the money in your bank account. Of course, if you only had $2, then 1 out of the 2 would be withdrawn. But, it's still considered one-half of your money even if you start with higher amounts. $50 out of $100 is one-half, $100 out of $200 is one-half, $24 out of $48 is one-half, and the list of fractions equivalent to $\frac{1}{2}$ is literally endless.

That is: $\frac{50}{100} = \frac{100}{200} = \frac{24}{48} = \frac{1}{2}$

$\frac{1}{2}$ is said to be in *simplest form* because of all of these fractions with the same amount, $\frac{1}{2}$ has the smallest denominator.

Note: A fraction in which the numerator is greater than the denominator is called an *Improper Fraction*, and they can be written in an equivalent form called a *Mixed Number*. For example, from above we saw that one and a quarter equivalent to five quarters. $1\frac{1}{4}$ is a mixed number, while $\frac{5}{4}$ is an improper fraction.

If a fraction has a numerator that is smaller than the denominator, it is called a *Proper Fraction*. $\frac{3}{4}$ is an example of a proper fraction. Proper fractions dont have a mixed number equivalent.

Exercise 4-7: Showing Equivalence

Write two other fractions equivalent to the given fraction. For the given fraction and each of your equivalent fractions, draw a shaded rectangle (or more than one if the given fraction is improper) showing that they all represent the same amount.

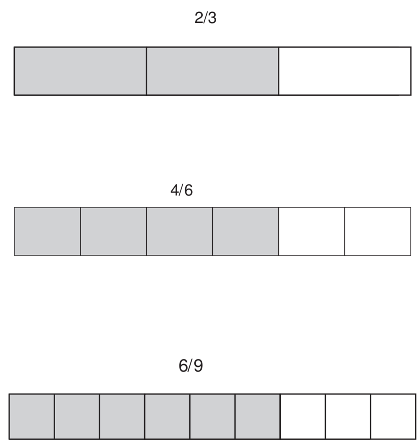
Example: $\frac{2}{3}$

Solution:

$\frac{2}{3}$

$\frac{4}{6}$

$\frac{6}{9}$



1. $\frac{1}{2}$

2. $\frac{1}{3}$

3. $\frac{2}{5}$

4. $\frac{3}{4}$

5. $\frac{3}{2}$

6. $\frac{1}{8}$

7. $\frac{3}{8}$

8. $\frac{4}{3}$

9. $\frac{7}{4}$

10. $\frac{6}{10}$

Adding Mixed Numbers: Adding mixed numbers is similar to adding whole numbers. The only difference is that the fraction parts of the numbers must be thought of as their own place value.

Example 1: Add $12\frac{3}{5} + 28\frac{2}{5} + 55\frac{4}{5}$

Solution: First estimate as usual: $10 + 30 + 60 = 100$ so that we know that the sum is approximately 100. For the actual sum, line up the tens, ones, and fifths for each number:

	Tens	Ones	Fifths
$12\frac{3}{5}$	1	2	3
$28\frac{2}{5}$	2	8	2
$55\frac{4}{5}$	5	5	4
Totals before carrying:	8	15	9
Carries:	1	1	not applicable
Answer:	9	6	4

which means $96\frac{4}{5}$. Notice that whole numbers carry to the next decimal place in packets of 10, while fractions carry over in packets with enough peices to make a whole. For example, since we had a total of 9 fifths, 5 of the fifths carried over to one extra whole, leaving 4 fifths left.

Exercise 4-8: Add the mixed numbers in the sums or solve the word problems. Write the answers as mixed numbers, with the fraction part in simplest form.

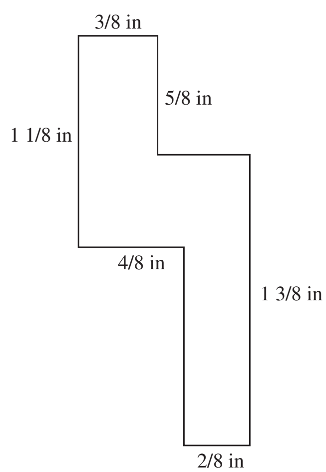
1. $2\frac{5}{8} + 7\frac{5}{8}$

2. $5\frac{3}{4} + 14\frac{3}{4} + 5\frac{3}{4} + 14\frac{3}{4}$

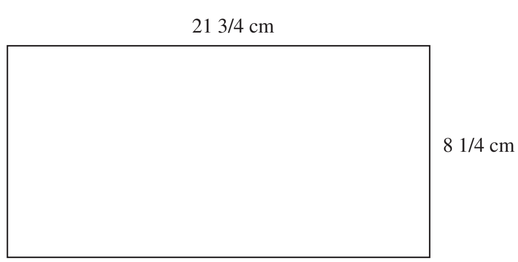
3. $1\frac{9}{16} + 4\frac{5}{16}$

4. For the four day weekend, you decide to do some baking. Your chocolate chip cookie recipe calls for $1\frac{1}{4}$ cups of sugar. The banana bread needs $\frac{3}{4}$ cups of sugar. The cake you are making from scratch calls for $2\frac{3}{4}$ cups of sugar. If you make a double batch of coockies, and three batches of banana bread, and one cake, how much sugar will you need?

5. Find the perimeter of the following figure:

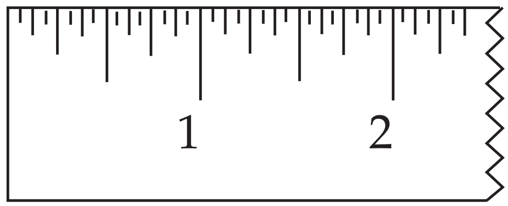


6. Find the perimeter of the following figure:

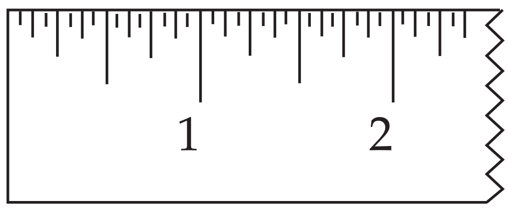


Exercise 4-9: Mark the following fractions of 1 inch on the given rulers.

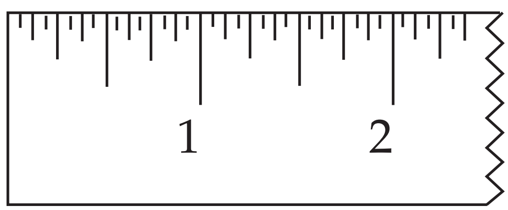
1. $\frac{1}{2}''$



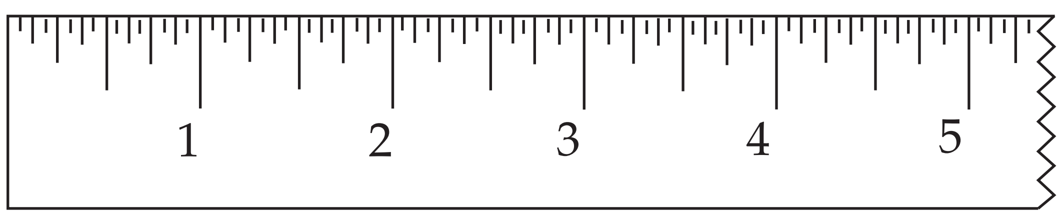
2. $\frac{3}{4}''$



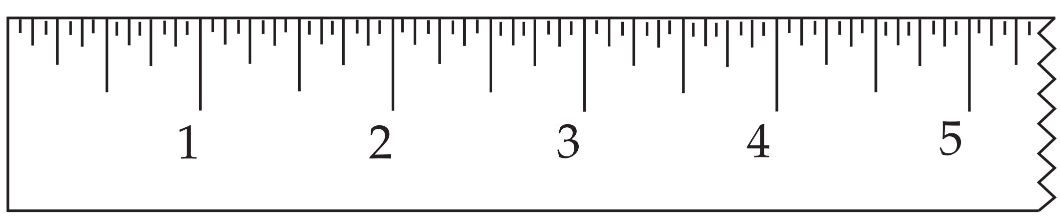
3. $1\frac{3}{8}''$



4. $3\frac{5}{8}''$



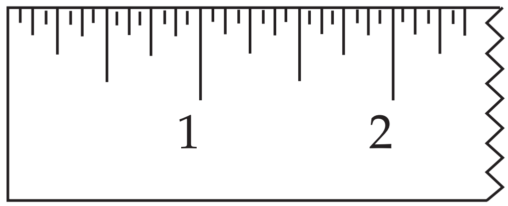
5. $2\frac{1}{16}''$

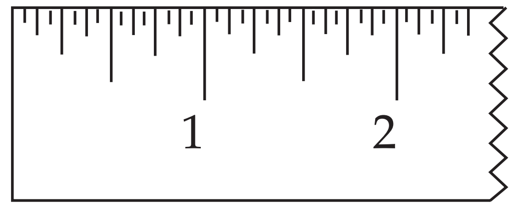


Exercise 4-10: 1. In each problem, add the fractions of an inch and mark them on the ruler below.

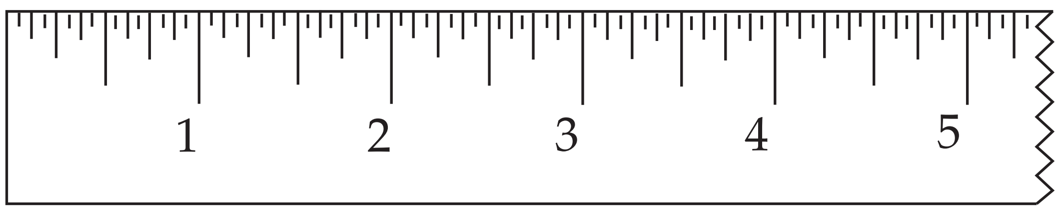
(a) $\frac{5}{8}'' + \frac{7}{8}'' =$ ____________

(b) $\frac{7}{16}'' + \frac{13}{16}'' =$ ____________

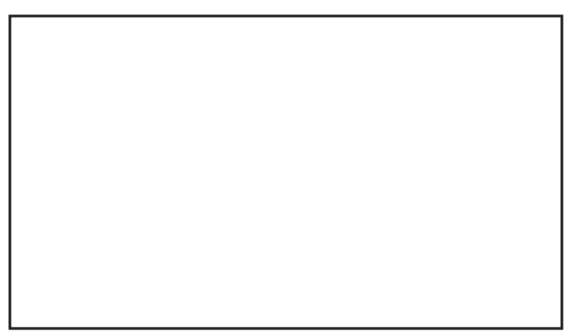




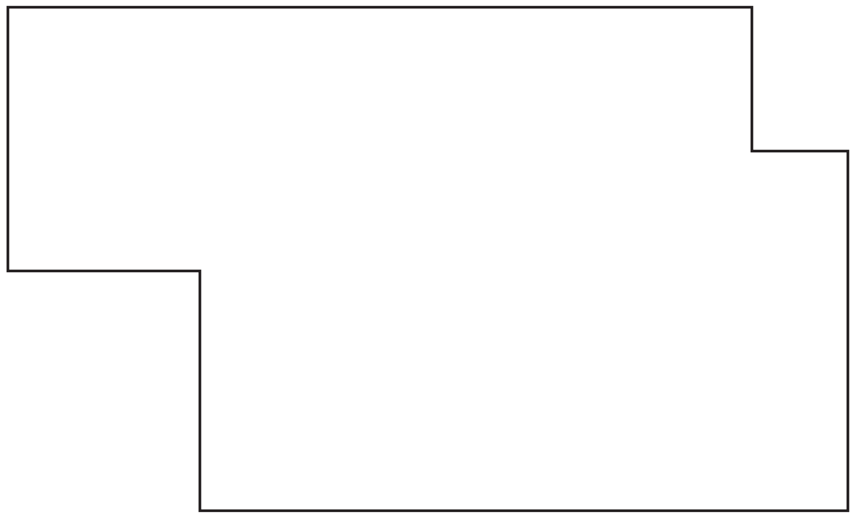
$2\frac{1}{2}'' + 1\frac{3}{4}'' =$ ____________



2. Using a ruler to measure the sides to the nearest eighth of an inch, confirm that the perimeter of the rectangle below is 9 inches. Mark the measurements on the figure.



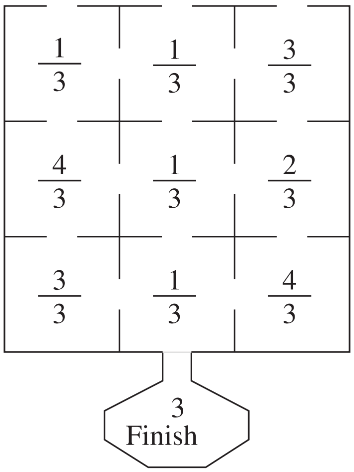
3. Use a ruler and measure the sides of the shape below to the nearest eighth of an inch in order to determine its perimeter. Mark the measurements that you use on the figure. (Hint: You may not need to measure every side!)



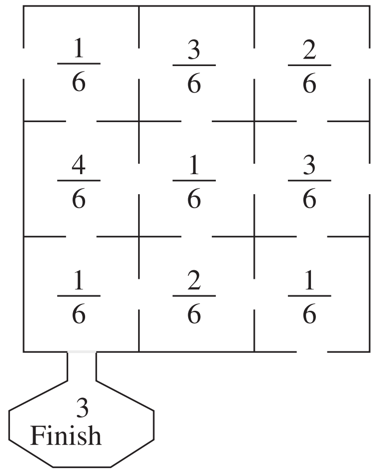
Fraction Paths

Start at any of the openings along the outside of the square and draw a path through the fractions that adds up to the number at the Finish.

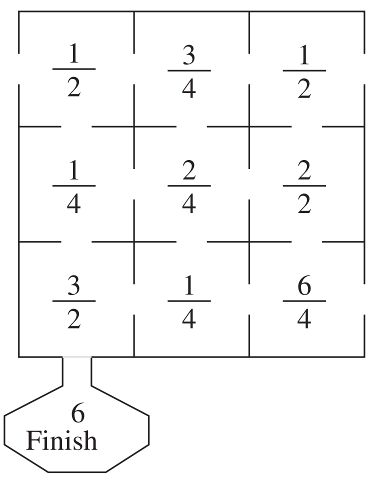
1.



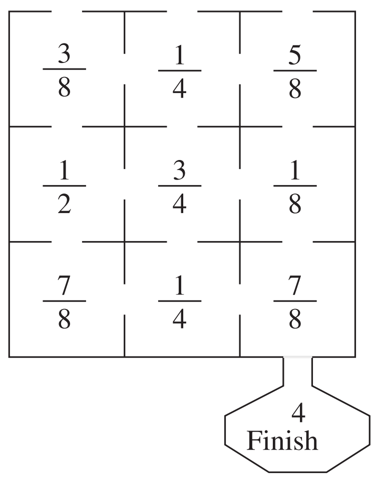
2.



3.



4.



5 Subtraction of Whole Numbers, Decimals, and Fractions

5.1 Money Activity:

Suppose the bank has a pile of bills in denominations $10,000, $1000, $100, $10, $1, dimes and pennies. There are three members in each group. Each member gets a turn being the banker, and loaning money to the other two. When each member takes their loan, make sure that the number of bills in each denomination is less than ten at any given time. When the banker gives out the loan, start at the lowest denomination, and work to the higher denominations.

1. Group member #1 is the banker. Begin with $985,347.10 to lend.

   (a) Group member #2 borrows $2,549.50 from the bank (member #1). In the table below, record the bank's initial balance, and new balance after the transaction.

   (b) Group member #3 borrows $752.66 from the bank. In the table below, record the bank's initial balance, and new balance after the transaction.

Bank's initial balance:	
Balance after member #2 loan:	
Balance after member #3 loan:	

2. Group member #2 is the banker. Begin with the balance left over from when member #1 was the banker.

   (a) Group member #1 borrows $2,395.48 from the bank (member #2). In the table below, record the bank's initial balance, and new balance after the transaction.

   (b) Group member #3 borrows $189.80 from the bank. In the table below, record the bank's initial balance, and new balance after the transaction.

Bank's initial balance:	
Balance after member #1 loan:	
Balance after member #3 loan:	

3. Group member #3 is the banker. Begin with the balance left over from when member #2 was the banker.

   (a) Group member #1 borrows $525.25 from the bank (member #3). In the table below, record the bank's initial balance, and new balance after the transaction.

   (b) Group member #2 borrows $298.12 from the bank. In the table below, record the bank's initial balance, and new balance after the transaction.

Bank's initial balance:	
Balance after member #1 loan:	
Balance after member #2 loan:	

4. Compare your results with another group in the class.

5. How would you compare this activity with the method for subtraction that you know?

5.2 The process of subtraction:

Please notice that every time you were borrowing money during the acticity, if the number of bills from any denomination was larger than the bank's number of bills from the same denomination, the bank had to make an exchange for equal amount of money with higher denomination bills. For example, assume bank had \$25 as two \$10 bills and five \$1 bills and you were borrowing \$8. The bank had to exchange one of its \$10 bills with ten \$1 bills to make its \$25 to be as one \$10 bill and fifteen \$1 bills. Now, to borrow eight \$1 billsthe bank has enough \$1's. The banks balance would be \$17, as one \$10 bill and seven \$1 bills.

We use the same idea of borrowing when we subtract any two numbers.

Vocabulary

- The operation is called **subtraction**.
- The number to subtract from is called the **minuend**.
- The number being subtracted is called the **subtrahend**.
- The result of the operation is called the **difference**.

For example, in $8 - 5 = 3$, the 8 is the minuend, the 5 is the subtrahend, and the 3 is the difference.

Steps for subtracting whole numbers and decimals:

1. As for addition, line up the numbers, digit by digit, so that each place value is in the same column. The number you are subtracting from (Minuend) should be on top, and the number you are subtracting (Subtrahend), below that.
2. Start from the right most column.
3. If the digit on top is greater than or equal to the digit on the bottom, take away the digit on the bottom from the digit on the top and write the difference below the bottom digit in the same column.
4. If the digit on top is smaller than the digit on the bottom, borrow from the digit to the left on top, giving you an extra ten, enough to take away. To indicate the extra ten, write a small 1 just to the left of the top number's digit. Then take away the bottom and write the difference below the bottom digit.

Example 1: $7,458.39 - 5,297.58$

Solution:

$$\begin{array}{ccccccccc} & 7 & , & 4 & 5 & 8 & . & 3 & 9 \\ - & 5 & , & 2 & 9 & 7 & . & 5 & 8 \\ \hline & & & & & & & & 1 \end{array}$$

$$\begin{array}{ccccccccc} & & & & & 7 & & & \\ & 7 & , & 4 & 5 & \not{8} & . & {}_{1}3 & 9 \\ - & 5 & , & 2 & 9 & 7 & . & 5 & 8 \\ \hline & & & & & & & 8 & 1 \end{array}$$

$$\begin{array}{cccccccccc}
 & & & 3 & & 7 & & & \\
 & 7 & , & \cancel{4} & {}^{1}5 & \cancel{8} & . & {}^{1}3 & 9 \\
- & 5 & , & 2 & 9 & 7 & . & 5 & 8 \\
\hline
 & & & & & 0 & . & 8 & 1
\end{array}$$

$$\begin{array}{cccccccccc}
 & & & 3 & & 7 & & & \\
 & 7 & , & \cancel{4} & {}^{1}5 & \cancel{8} & . & {}^{1}3 & 9 \\
- & 5 & , & 2 & 9 & 7 & . & 5 & 8 \\
\hline
 & 2 & , & 1 & 6 & 0 & . & 8 & 1
\end{array}$$

So, the difference of 7,458.39 and 5,297.58 is 2,160.81. Granted, with all of the markings for the borrowing, the process can get pretty messy. If your handwriting is messy already, try writing bigger. Everyone needs to be careful to keep the place value columns in line.

One nice advantage to subtraction, is that you can check your answer with addition. If your difference is correct, then the difference plus the subtrahend will equal the minuend. In our case, to check, we see that $2,160.81 + 5,297.58 = 7,458.39$

$$\begin{array}{ccccccccc}
 & & & 1 & & 1 & & & \\
 & 2 & , & 1 & 6 & 0 & . & 8 & 1 \\
+ & 5 & , & 2 & 9 & 7 & . & 5 & 8 \\
\hline
 & 7 & , & 4 & 5 & 8 & . & 3 & 9
\end{array}$$

Example 2: $24,215.4 - 35.17$

Solution:
Notice the extra zero in the minuend as a place-holder for the hundredths place.

$$
\begin{array}{cccccccccc}
 & & & & & & & 3 & \\
 & 2 & 4 & , & 2 & 1 & 5 & . & \cancel{4} & {}^{1}0 \\
- & & & & & 3 & 5 & . & 1 & 7 \\
\hline
 & & & & & & & & & 3
\end{array}
$$

$$
\begin{array}{cccccccccc}
 & & & & 1 & & & & 3 & \\
 & 2 & 4 & , & \cancel{2} & {}^{1}1 & 5 & . & \cancel{4} & {}^{1}0 \\
- & & & & & 3 & 5 & . & 1 & 7 \\
\hline
 & & & & & & 0 & . & 2 & 3
\end{array}
$$

$$
\begin{array}{cccccccccc}
 & & & & 1 & & & & 3 & \\
 & 2 & 4 & , & \cancel{2} & {}^{1}1 & 5 & . & \cancel{4} & {}^{1}0 \\
- & & & & & 3 & 5 & . & 1 & 7 \\
\hline
 & 2 & 4 & , & 1 & 8 & 0 & . & 2 & 3
\end{array}
$$

So, the difference of 24,215.4 and 35.17 is 24,180.23. Use addition to check:

$$
\begin{array}{cccccccccc}
 & & & & 1 & & & & 1 & \\
 & 2 & 4 & , & 1 & 8 & 0 & . & 2 & 3 \\
+ & & & & & 3 & 5 & . & 1 & 7 \\
\hline
 & 2 & 4 & , & 2 & 1 & 5 & . & 4 & 0
\end{array}
$$

Exercise 5-1: Find the following defferences to practice subtraction.

1. $236,972.112 - 35,387.17$

2. $76,002.175 - 12,378.1892$

3. $38,937.08 - 12,609.5$

4. $237.0116 - 59.983$

5.3 Alternate subtraction method:

Subtraction by Counting Up

When we subtract using the standard method introduced in the previous section, we get a result that looks something like the example below.
Example:

$$\begin{array}{rcc} & 5 & 4 \\ - & 2 & 9 \\ \hline \end{array} \qquad \longrightarrow \qquad \begin{array}{rcc} & {}^{4}\not{5} & {}_{1}4 \\ - & 2 & 9 \\ \hline & 2 & 5 \end{array}$$

There are two important things to notice about the result:

(1) We can check our work by adding the bottom two numbers to see if we get the top number (does $25 + 29 = 54$?).

(2) This gives us an alternative way of subtracting numbers. Since the result of subtraction (the difference) adds with the lower number (the subtrahend) to make the top number, we can figure out the difference by counting up from the bottom to the top. For example, let's use this approach with the example above:

$$\begin{array}{rcc} & 5 & 4 \\ - & 2 & 9 \\ \hline \end{array} \quad \xrightarrow{+1} \quad 30 \quad \xrightarrow{+20} \quad 50 \quad \xrightarrow{+4} \quad 54$$

By adding 1 to 29 we jump to 30, a nice round number, and a relatively easy number to jump to 50 from - just add 20. Once we're at 50 we only have to add 4 more to get to 54. If we add up the numbers we used to make our jumps: $1 + 20 + 4 = 25$, we get the difference.
Try the next example on your own before reading the solution:

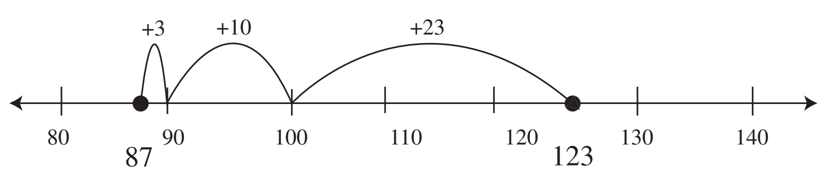
$$\begin{array}{rccc} & 1 & 2 & 3 \\ - & & 8 & 7 \\ \hline \end{array}$$

Solution:

$$\begin{array}{rccc} & 1 & 2 & 3 \\ - & & 8 & 7 \\ \hline \end{array} \quad \xrightarrow{+3} \quad 90 \quad \xrightarrow{+10} \quad 100 \quad \xrightarrow{+23} \quad 123$$

Adding the jumps gives us $3 + 10 + 23 = 36$ so we know that $123 - 87 = 36$.

The graph below shows what these jumps look like on a number line.



These examples offer one possible way to count up from one number to another. You might have chosen different jumps than those shown here and so long as you add your jumps correctly when you count them up, the jumps you choose are up to you.

Continue finding the differences. Use the alternate method.

5. $12.3861 - 8.0998$

6. $892,543.09 - 876,119.112$

7. $6,321,311.061 - 99,378.5$

Find the last three differences using either method.

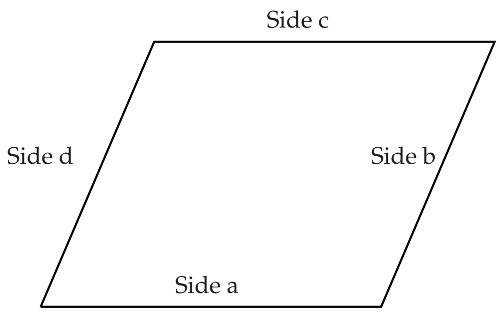
8. $3,012.9 - 2,981.16$

9. $6,429,199 - 399.05$

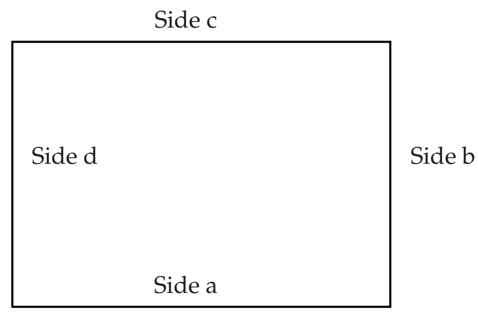
10. $12,001,260.01 - 10,652,981.7$

Fencing Activity

Joe and Kathy are neighbors who are fencing their backyards in the following shapes:



Joe's back yard



Kathy's back yard

1. Measure each side of the pictures of Joe's and Kathy's backyards in centimeters. Round each measurement to the nearest centimeter, and record it in the table. Assume that every centimeter on the picture represents one yard in real life. Record the real life distances. Be sure to write units for all of your values!

Side	Joe's picture	Joe's actual	Kathy's picture	Kathy's actual
a				
b				
c				
d				

2. How many yards of fencing does Joe need?

3. How many yards of fencing does Kathy need?

4. Who needs more fencing, Joe or Kathy?

5. How much more do they need?

Find The Digits

Fill in the blanks to complete the addition and subtraction problems below.

1.

$$\begin{array}{rccc} & 5 & 1 & 8 \\ + & 3 & \square & 3 \\ \hline & \square & 6 & 1 \end{array}$$

2.

$$\begin{array}{rccc} & 1 & 7 & 3 \\ & + & \square & 4 \\ \hline & 2 & 6 & 7 \end{array}$$

3.

$$\begin{array}{rccc} & \square & 2 & 7 \\ + & 5 & 9 & \square \\ \hline 1 & 5 & \square & 1 \end{array}$$

4.

$$\begin{array}{cccc} 1 & 2 & \square & 2 \\ - & 4 & 3 & 3 \\ \hline & 8 & 5 & \square \end{array}$$

5.

$$\begin{array}{ccc} 1 & \square & 6 \\ - & 5 & \square \\ \hline \square & 3 & 4 \end{array}$$

6.

$$\begin{array}{rcccc} & 2 & 5 & 3 & 2 \\ - & \square & 8 & 1 & \square \\ \hline & & \square & 1 & 6 \end{array}$$

7.

$$\begin{array}{cccc} & 6 & 5 & \square \\ + & 8 & \square & 5 \\ \hline 1 & 5 & 0 & 8 \end{array}$$

8.

$$\begin{array}{rccc} & 6 & 4 & \square \\ - & \square & \square & 8 \\ \hline & 3 & 4 & 9 \end{array}$$

9.

$$\begin{array}{cccc} \square & 0 & 3 & 5 \\ - & 6 & 3 & \square \\ \hline & 3 & \square & 8 \end{array}$$

10.

$$\begin{array}{rccc} & 5 & 7 & \square \\ & & \square & 6 \\ + & 2 & 4 & 3 \\ \hline & \square & 9 & 6 \end{array}$$

11.

$$\begin{array}{rcccc} & & 8 & \square & 2 \\ & & \square & 4 & 6 \\ + & & & & \square \\ \hline & \square & 3 & 8 & 4 \end{array}$$

12.

$$\begin{array}{rcccc} & \square & \square & 5 & 3 \\ & & 5 & 3 & 7 \\ + & & 6 & \square & 2 \\ \hline & 4 & 8 & 9 & \square \end{array}$$

Find The Patterns

For the following lists of numbers,

- Describe the pattern in words.
- Fill in the missing numbers.

1. $2, 4, 6, 8, \_\_\_\_, 12, \ldots$

2. $10, 13, 16, \_\_\_\_, 22, 25, \ldots$

3. $28, 23, \_\_\_\_, \_\_\_\_, 8, 3$

4. $\frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \_\_\_\_, 1\frac{3}{8}, 1\frac{5}{8}, \_\_\_\_ \ldots$

5. $32.7, 32.3, \_\_\_\_, \_\_\_\_, 31.1, \ldots$

6. $23, \_\_\_\_, 35, \_\_\_\_, 47, \_\_\_\_, 59, \ldots$

7. $\frac{1}{2}, 1\frac{1}{4}, \_\_\_\_, 2\frac{3}{4}, \_\_\_\_, \ldots$

5.4 Estimation or Approximation

The Process of Estimating with Subtraction:

- Round each number in the estimation problem to one significant digit.
- Subtract the numbers from step 1.

Example 1: Estimate the difference:

$7,649 - 365$

Solution:

7,649 rounded to one significant digit is 8,000

365 rounded to one significant digit is 400

The difference of 8000 and 400 is 7600. (Alternate method: Starting at 400, jump 100 to 500, jump another 500 to 1000, jump another 7000 to 8000. Estimated difference is the total jump of 7000+500+100=7600.)

The estimate of the difference of 7,649 and 365 is 7600.

Example 2: Estimate the difference:

$159.99 - 20.898$

Solution:

159.99 rounded to one significant digit is 200

20.898 rounded to one significant digit is 20

The difference of 200 and 20 is 180.

The estimate of the difference of 159.99 and 20.898 is 180.

Exercise 5-2: Approximate Difference Practice

For each of the following differences:

- Estimate the difference by rounding each number to one significant digit before subtracting.
- Find the actual difference.
- State whether the actual difference is $>$, $=$, or $<$ the estimate.

1. $45,467.9 - 128.4$

2. $890.13 - 632.9$

3. $70.5 - 19.12$

4. $3,749,221.11 - 67,921.5$

5. $2,039 - 1934.5$

6. $11,012 - 9,261$

7. $12,135,295 - 2,367,109$

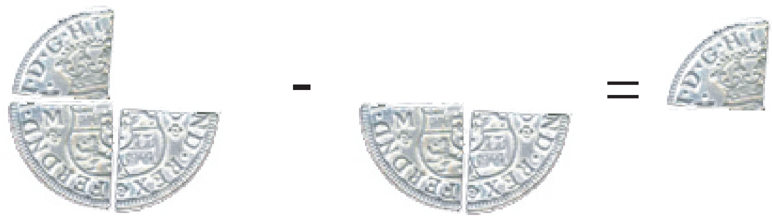
8. $17,726.14 - 17,632$

9. $10,138.22 - 8,924$

10. $1.0356 - 0.284$

5.5 Subtracting Fractions with Like Denominators

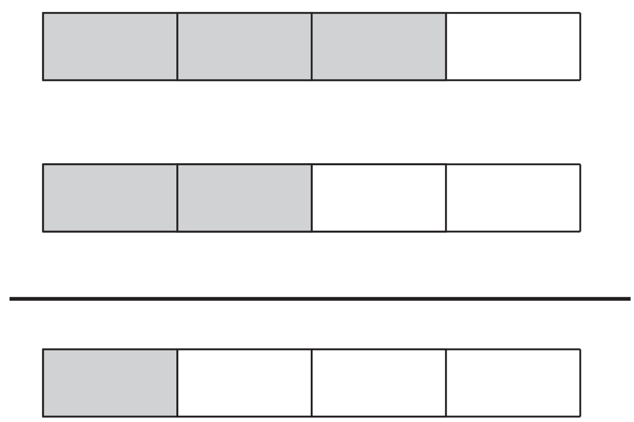
The idea of subtracting fractions is the same as with addition. For example, if you have three quarters and take away two quarters, you have one quarter left. Back in the early days that would look like:



$$\frac{3}{4} - \frac{2}{4} = \frac{1}{4}$$

With rectangles representing each whole, a representation of this problem might look like:

$$\frac{3}{4} - \frac{2}{4} = \frac{1}{4}$$



Exercise 5-3:
For each of the following differences:

- Draw a picture representing the the difference.
- Find the difference.

1. $\frac{2}{3} - \frac{1}{3}$

2. $\frac{5}{8} - \frac{3}{8}$

3. $\frac{3}{2} - \frac{1}{2}$

4. $\frac{5}{4} - \frac{3}{4}$

5. $\frac{7}{8} - \frac{1}{2}$

Some of the answers in the last exercise could be written as an *equivalent fraction* with a smaller denominator. For the problems in which this is true, check "yes", and write the original difference as well as the simpler, equivalent fraction. If there is no simpler, equivalent fraction, check "no".

Problem	Yes	No	Original Fraction	Equivalent Fraction
1				
2				
3				
4				
5				

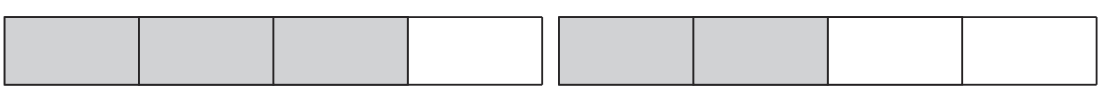
5.6 Comparing Fractions

How can we decide which fraction is bigger?

- If the fractions have the same denominator, then the size of the pieces is the same, and we are just comparing the number of pieces.

  Example 1:
  We can compare $\frac{3}{4}$ and $\frac{2}{4}$ by seeing that $\frac{3}{4}$ has one more fourth than $\frac{2}{4}$ does:

$$\frac{3}{4} > \frac{2}{4}$$

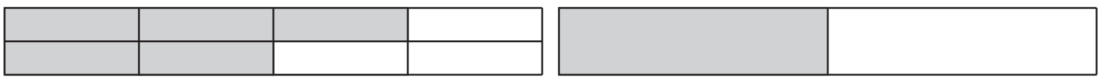


- If the fractions **do not** have the same denominators, we need to find equivalent fractions for each that **do** have the same denominator:

  Example 2:

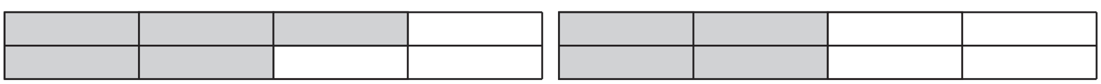
  How can we compare $\frac{5}{8}$ and $\frac{1}{2}$? By seeing that $\frac{1}{2}$ is equivalent to $\frac{4}{8}$, we can see that $\frac{5}{8}$ is bigger:

$$\frac{5}{8} \qquad ? \qquad \frac{1}{2}$$



ahhh...

$$\frac{5}{8} \qquad > \qquad \frac{4}{8}$$



Exercise 5-4:

For the following pairs of fractions,

- Draw a picture for each to figure out which is bigger.
- Place a $>$, $=$, or $<$ symbol between the fractions to indicate their relative size.

1. $\frac{3}{4}$ and $\frac{1}{4}$

2. $\frac{5}{8}$ and $\frac{7}{8}$

3. $1\frac{1}{3}$ and $\frac{2}{3}$

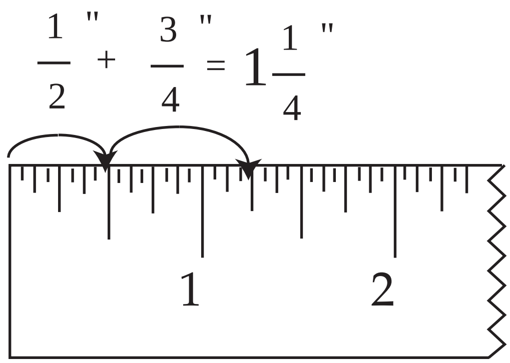
4. $\frac{3}{4}$ and $\frac{3}{2}$

5. $\frac{1}{2}$ and $\frac{3}{8}$

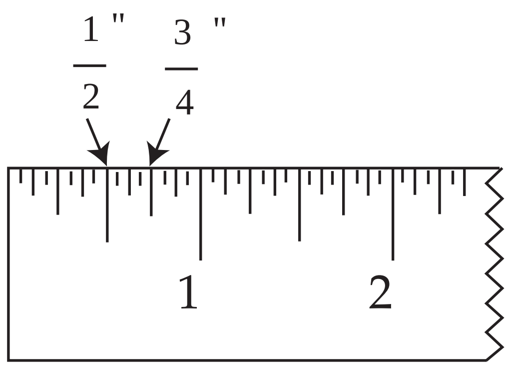
6. $\frac{1}{4}$ and $\frac{3}{8}$

7. $\frac{3}{4}$ and $\frac{5}{8}$

Another way to visualize which fraction is bigger is to think of them as lengths. Recall that when we were using the rulers to visualize the addition of fractions, we identified where the first fraction sat on the ruler, and then added the second fraction by jumping ahead a length equivalent to the other fraction. For example, to illustrate $\frac{1}{2} + \frac{3}{4}$, we drew a picture similar to the following:



To compare, just place each fraction on the ruler, and see which one is a longer length!

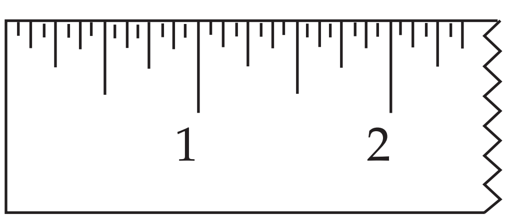


Exercise 5-5:

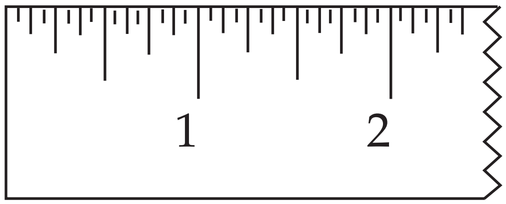
For the following pairs of fractions,

- Indicate where each fraction lies on the number line to see which is bigger.
- Place a $>$, $=$, or $<$ symbol between the fractions to indicate their relative size.

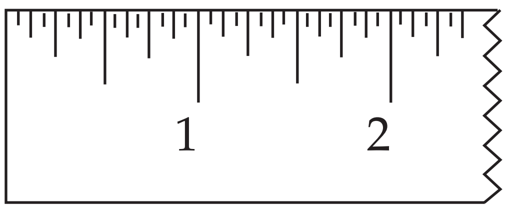
1. $1\frac{1}{8}$ and $1\frac{1}{4}$



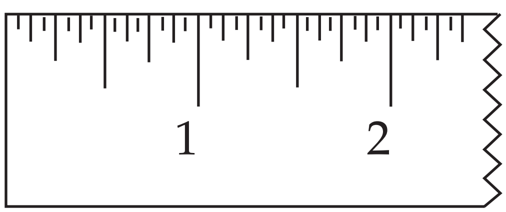
2. $\frac{3}{4}$ and $\frac{3}{8}$



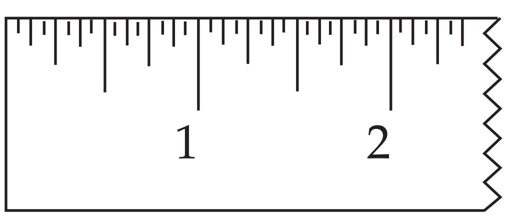
3. $\frac{5}{16}$ and $\frac{3}{8}$



4. $\frac{3}{4}$ and $\frac{7}{8}$



5. $\frac{1}{2}$ and $\frac{7}{16}$



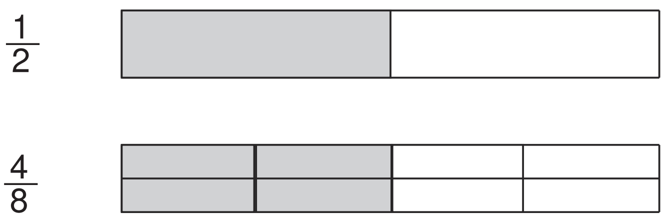
5.7 Subtraction in Real Life

One of the first places you will encounter subtraction after this class is...in your next math class! The most common words that are used to imply subtraction in math problems are "difference between" and "subtracted from". Because the order that numbers are subtracted changes the difference (unlike addition in which order doesn't matter), it is important to know what order is implied.

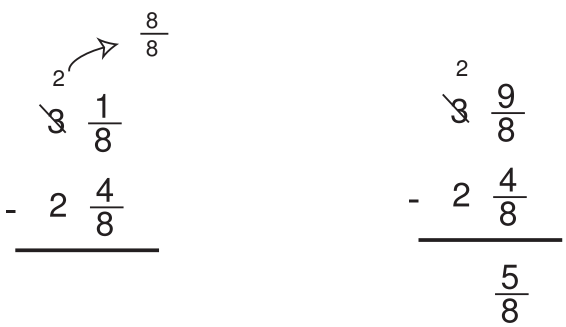
Example 1: What is the difference between $3\frac{1}{8}$ and $2\frac{1}{2}$?

Solution:

We can find the difference by subtracting $3\frac{1}{8} - 2\frac{1}{2}$. From previous problems and pictures, we have seen that $\frac{1}{2}$ is equivalent to $\frac{4}{8}$.



So, our difference is equivalent to $3\frac{1}{8} - 2\frac{4}{8}$. Because 4 eighths is more than 1 eighth, we have to borrow 8 eighths.



So, the difference between $3\frac{1}{8}$ and $2\frac{1}{2}$ is $\frac{5}{8}$.

Example 2: What is 179.95 subtracted from 438.50?

Solution:

When the words "subtracted from" are used, the order that the numbers appear is opposite the order that they are subtracted. 179.95 subtracted from 438.50 is equivalent to the difference between 438.50 and 179.95 which is 258.55. (Verify this! Math books often leave out details in an example if it's something that they think you have already done. For practice and to make sure, you should always go through on your own any details that are left out.)

So, 179.95 subtracted from 438.50 is 258.55.

In a more real life setting, there are three common situations where subtraction is used. They are finding a **difference** in two quantities, finding **how much more** you need, and finding the **change** in a quantity.

Difference: If you want to know how far apart two quatities are, literally find the **difference** in height, weight, price, bank balance, or anything that has size, subtract the smaller quatity from the larger quantity.

Example 1: At their 5 year check-up, the doctor weighed each twin. The older twin weighed 42.3 lbs, while the younger twin weighed 33.5 lbs. How much heavier is the older twin than the younger twin?

Solution: It is common in math problems, that the idea is the same as something you've studied, but the words used to describe the idea are different. You must practice the process of understanding the given words, by reading the passage many times and thinking clearly about what it *means*, then thinking of what familiar math idea that it is equivalent to. Only through practice, and asking questions if a certain problem doesn't make sense, will you get good at this. In this case, although the word "difference" is not used in the problem, the given question is equivalent to the question, "What is the difference in their weights?". The difference in their weights is $42.3 - 33.5 = 8.8$, so,

the older twin is 8.8 lbs heavier than the younger twin.

If you have a certain amount of something and want to know **how much more** you need to get to a total, subtract how much you have from the total that you want. The difference is how much more you need.

Example 2: A math teacher is saving to buy a 2002 Prius 4-door which costs $15,998.95. He has already saved $6,362.17. How much more does he need to save?

Solution: This time the wording in the problem is identical to our category. The required amount to save is how much he has saved already, $6,362.17, subtracted from the total that he needs for the car, $15,998.95. $15,998.95 - 6,362.17 = 9,636.78$, so,

he needs to save $9,636.78 more.

The final common use for subtraction that we will discuss here, is finding the **change** in a quatity. Whether something gets bigger or smaller, the change is the bigger value minus the smaller value.

Example 3: The percentage of young adults (age 18 to 24 years) who voted in various presidential elections are shown in the table:

Year	Percent
1972	50
1980	40
1984	41
1988	36
1996	32
2000	26

What was the change in the percentage of young adults voting between the years 1980 and 2000?

Solution: Here is a time where we use the identification numbers 1980 and 2000, to look up the values we need for the calculation. The percentage changed from 40 to 26 during that time, so the change is $40 - 26 = 14$. So,

the percentage changed by 14%.

Exercise 5-6: Answer the following problems. Use addition or subtraction as appropriate.

1. Find the difference between $5\frac{3}{8}$ and $4\frac{3}{4}$.

2. On Tuesday, the high temperature in Pacifica was 58.7°F. By Saturday, the high temperature was 64.9°F. How much hotter was it on Saturday than on Tuesday?

3. In preparing dinner for the guests, Billy needed $2\frac{1}{2}$ cups of chicken broth for the soup, and another $\frac{3}{4}$ cups of chicken broth for the pasta sauce. How much chicken sauce does he need to prepare the dinner?

4. Jing-jing was putting up a shelf in her closet. She needed a 2-by-4 that was $5\frac{5}{8}$ feet long, but the lumber yard sold her a 2-by-4 that was 7 feet long. How much does she have to cut off to make the board the correct length?

5. What is 14,375.04 subtracted from 32,191.23?

6. Paul was working the concessions stand at the basketball game. A customer ordered a popcorn for \$3.50, a hotdog for \$4.25, a soda for \$0.90, and a candy bar for \$0.85. The customer gave Paul a 20 dollar bill. How much is the change?

7. The balance in Hector's checking account was \$134.72 before the deposit. After the deposit, the balance was \$473.28. How much was the deposit?

8. Geoff saws $3\frac{7}{8}$ feet off of a branch that measured $10\frac{3}{16}$ feet. How long is the branch after Geoff makes the cut?

9. In 1993 the number of deaths in the U.S. due to the AIDS virus was 45,381. In 1999, the number of AIDS deaths in the U.S. was down to 16,273. What was the change in number of AIDS deaths?

10. Garret wanted to try to bake bread. The recipe called for $4\frac{1}{2}$ cups of flour, but when he looked in the cupboard, he saw that he only had $2\frac{1}{8}$ cups left. How much more flour does he need?

6 Multiplication of Whole Numbers and Decimals

6.1 Money Activity:

Suppose the bank has a pile of bills in denominations \$10,000, \$1000, \$100, \$10, \$1, dimes and pennies. There are three members in each group.

1. Each member in the group takes \$2,739.20.

   (a) Collect all three member's money in one pile, and record the total number of bills of each denomination in the table.

\$10,000	\$1,000	\$100	\$10	\$1	\$0.10	\$0.01

   (b) Now, do the necessary bill exchange in your total to keep the number of bills in each denomination less than ten. Record the number of bills in each denomination as well as your total value of all of the money.

\$10,000	\$1,000	\$100	\$10	\$1	\$0.10	\$0.01	Total \$

2. Start over and this time each member in the group takes \$675.95.

   (a) Collect all three member's money in one pile, and record the total number of bills of each denomination in the table.

\$10,000	\$1,000	\$100	\$10	\$1	\$0.10	\$0.01

   (b) Now, do the necessary bill exchange in your total to keep the number of bills in each denomination less than ten. Record the number of bills in each denomination as well as your total value of all of the money.

\$10,000	\$1,000	\$100	\$10	\$1	\$0.10	\$0.01	Total \$

3. Describe the two different arithmetic operations that you could have used to find the totals in questions (1) and (2).

4. Perform the following multiplications. You may use the money to help.

   (a) $(346.14)(4)$

   (b) $6,380.12 \times 5$

   (c) $278.29 \cdot 7$

   (d) $12(0.25)$

5. Practice muliplying by powers of ten by thinking of money.

   (a) What is the value of one hundred $5 bills?

   (b) How much is ten $20 bills worth?

   (c) How much is one hundred quarters worth?

   (d) One dollar is equal to how many dimes? How many pennies?

6. How much is twice $36.19?

7. If you triple $45.25, how much do you have?

Complete the following multiplication table table:

×	1	2	3	4	5	6	7	8	9	10	11	12
1												
2												
3												
4												
5												
6												
7												
8												
9												
10												
11												
12												

Fill in the top-most row and left-most column with the numbers 1 through 12, but not in order. Then fill out the multiplication table:

×												

Exercise 6-1:

1. Complete the multiplication table below.

$\times$	10	100	$1,000$	$10,000$	$100,000$
a) 8					
b) 37					
c) 409					
d) 1471					
e) 5040					
f) 0.3					
g) 0.047					
h) 1.06					
i) 14.109					
j) 10.7083					

2. Remember that multiplication is repeated addition so $3 \times 5 = 5 + 5 + 5 = 15$. Explain the meaning of 10×67.

3. Multiply the following in your head:

a) $50 \times 7 =$ ____________ b) $30 \times 80 =$ ____________ c) $60 \times 110 =$ ____________

d) $500 \times 70 =$ ____________ e) $4000 \times 900 =$ ____________

f) $2000 \times 7000 =$ ____________ g) $1200 \times 700 =$ ____________

4. Estimate the products below by first rounding to one significant digit.

Example:

$$\begin{array}{r} 23 \\ \times\ 58 \\ \hline \end{array} \longrightarrow \begin{array}{r} 20 \\ \times\ 60 \\ \hline 1200 \end{array}$$

a) $\begin{array}{r} 67 \\ \times\ 115 \\ \hline \end{array}$

b) $\begin{array}{r} 367 \\ \times\ 639 \\ \hline \end{array}$

c) $\begin{array}{r} 548 \\ \times\ 971 \\ \hline \end{array}$

d) $\begin{array}{r} 451 \\ \times\ 3501 \\ \hline \end{array}$

For the following problems, figure out the answer in two ways. Show your steps.

(a) Using repeated addition.
(b) Using multiplication.

5. The math club was ready to buy t-shirts. Each shirt cost $12. If there are 8 members in the club who want to buy shirts, what is the total cost for the shirts?

(a) Repeated addition:

(b) Multiplication:

6. While driving to L.A., Elissa averaged 60 miles per hour. She traveled for 3 hours at this speed before taking a rest break. How far had she traveled before taking the break?

   (a) Repeated addition:

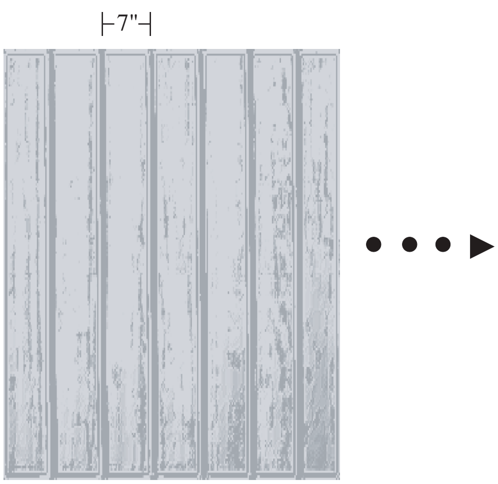
   (b) Multiplication:

7. A classroom has 5 rows of desks. Each row has 8 desks in it. How many desks are in the classroom?

   (a) Repeated addition:

   (b) Multiplication:

8. Max is building a fence. Each board is 7″ wide, and he has 64 boards. How long is the longest fence he can build? (See picture)



   (a) Repeated addition:

   (b) Multiplication:

Area Activity

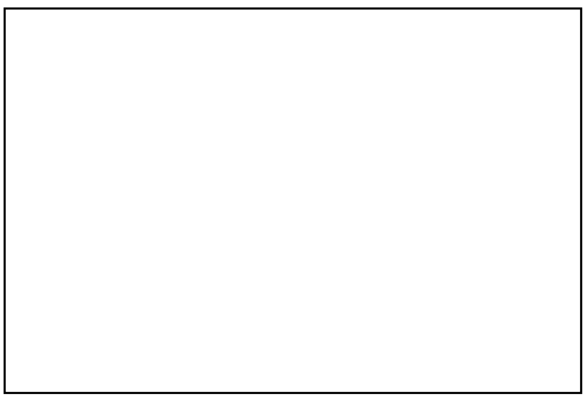
Using your square inches in your groups, estimate the areas of the following:

1. The cover of a binder or notebook or portfolio.

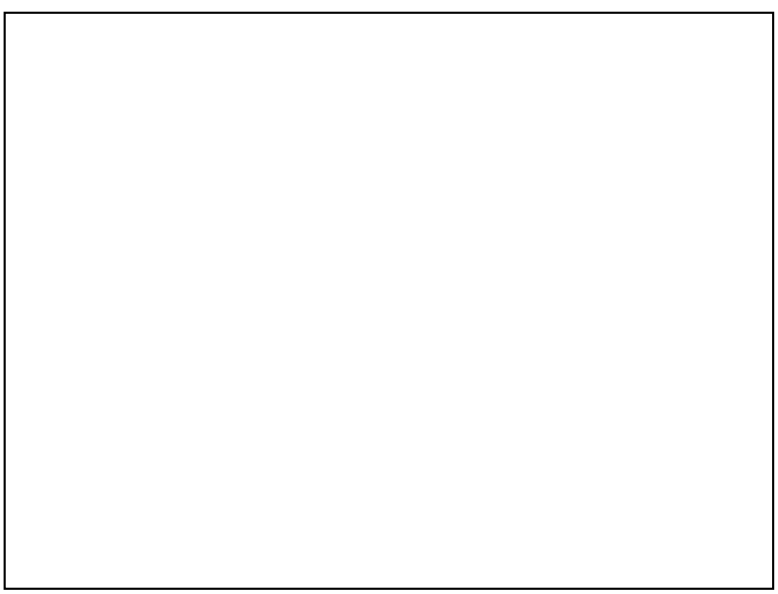
2. The top of a desk.

3. The back of a calculator.

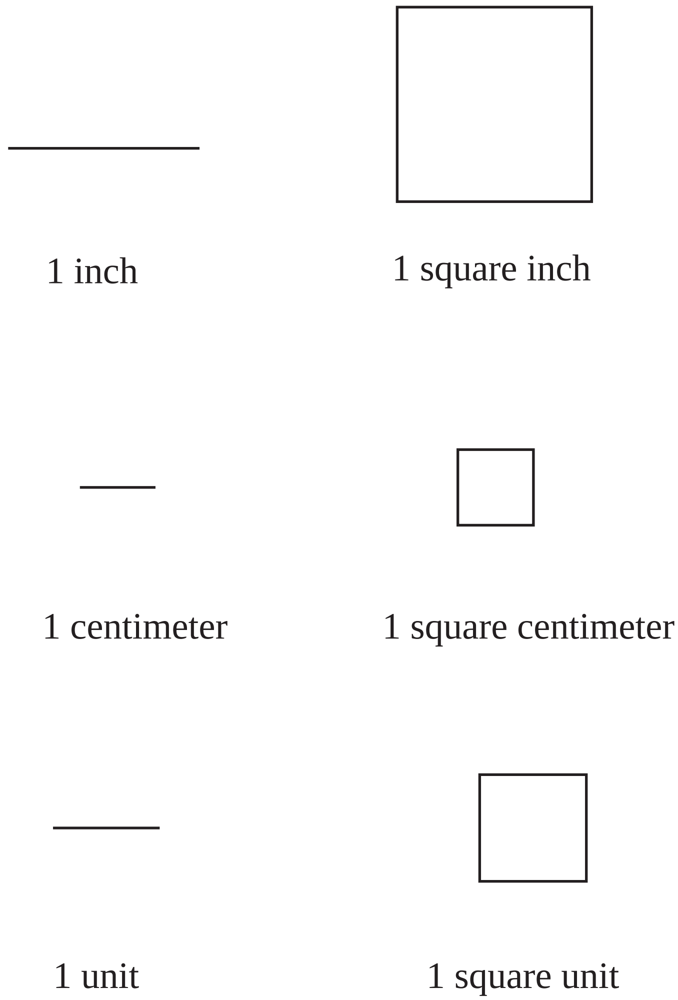
4. The following rectangle.



5. By using you ruler instead of the square inch pieces, find the area of the following rectangle to the nearest square inch. Describe your proceedure.

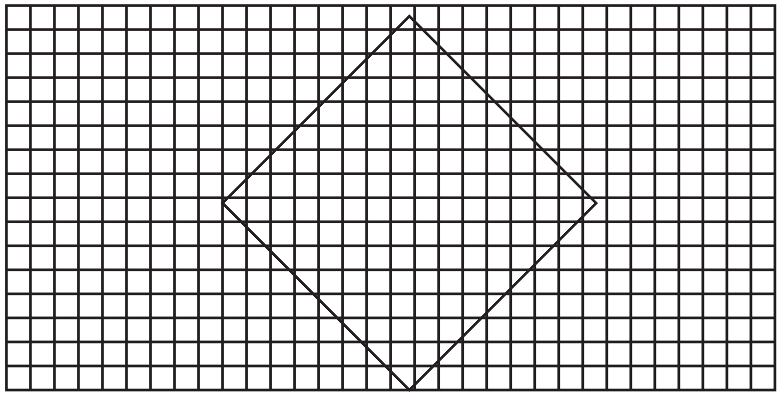


In the previous activity we saw that **area** is the measure of how many squares fit in an enclosed flat space. In the activity, you used square pieces that measured 1 inch on each side, so the units of the measurements were called **square inches**. One can measure area with square inches, square centimeters, square miles, or even square units, as long as the length of the units are defined.

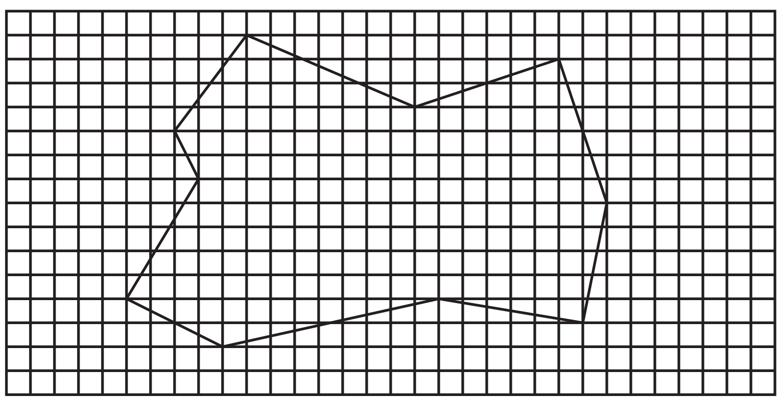


Exercise 6-2: For the following shapes, estimate the areas in square units given that each unit is ___ , and each square unit is □.

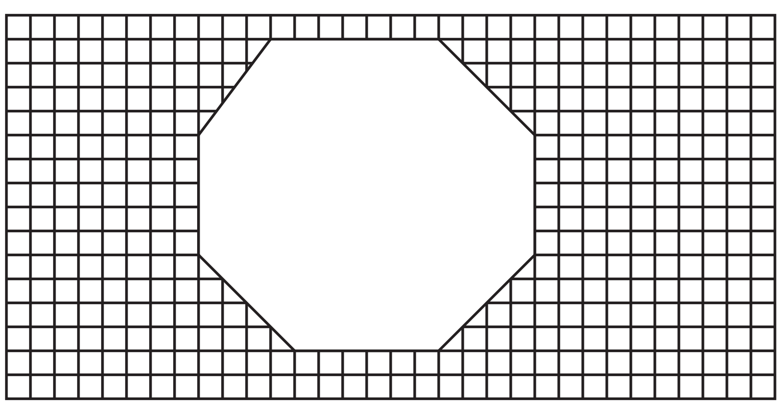
1.



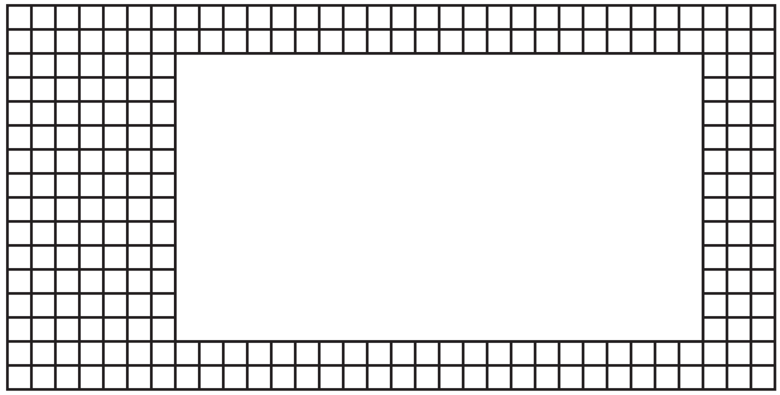
2.



3.



4.



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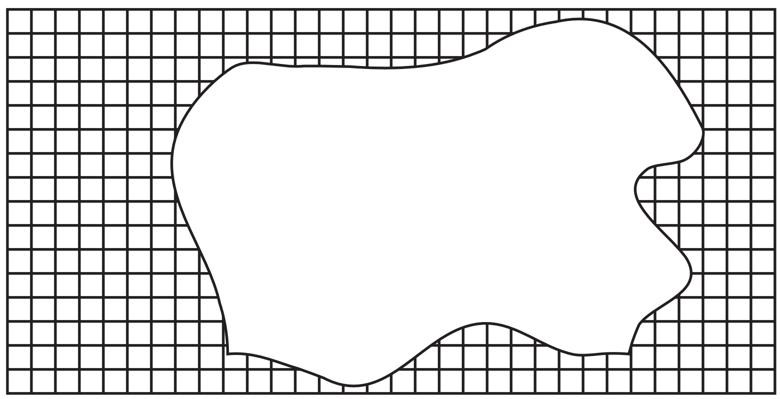

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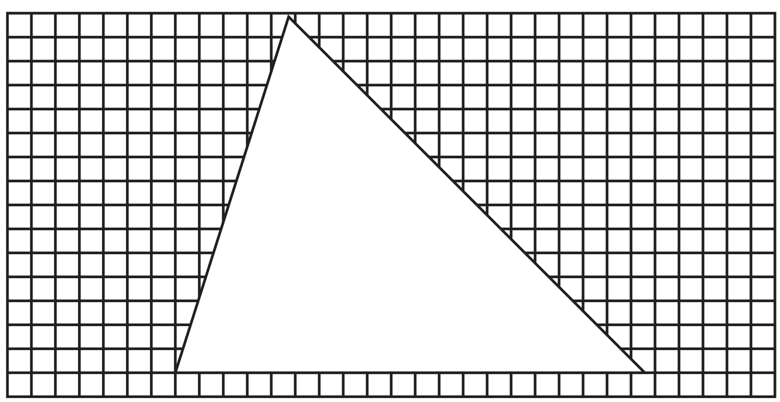
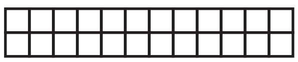


Area Formula For a Rectangle

In order to find the area of a shape without using square tiles and counting, it is important to understand how repeated addition or multiplication is involved in the area of a rectangle. If you have a very thin rectangle that is only 1 unit wide, then finding out the area in square units is simply counting how many squares are in that one row:

12 square units

The wider the rectangle gets, the more rows you have.

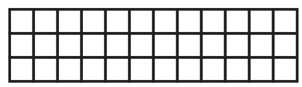


12 + 12 =

12 two times =

$2 \times 12 =$

24 square units

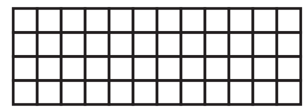


12 + 12 + 12 =

12 three times =

$3 \times 12 =$

36 square units



12 + 12 + 12 + 12 =

12 four times =

$4 \times 12 =$

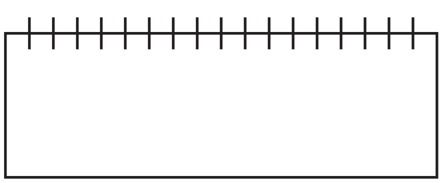
48 square units

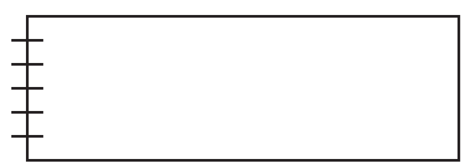
So, to figure out the area of a rectangle, you just need to know how many squares in each row, and how many rows. To find out how many squares in each row, measure how many units fit across the rectangle, and to find out how many rows fit, measure how many units fit down the rectangle.

Example 1:

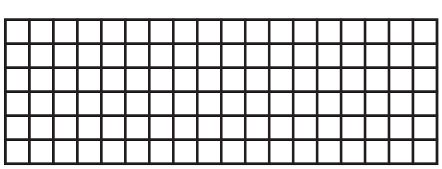
Find the area in square units of the following rectangle, if each unit is ___ , and each square unit is □.

Solution: The rectangle measures 18 units across and 6 units down.





So, the area of the rectangle is $18 + 18 + 18 + 18 + 18 + 18 = 6 \times 18 = 108$ square units.

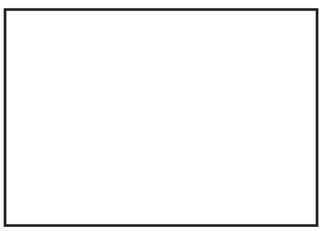


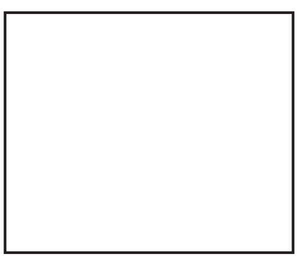
Exercise 6-3:

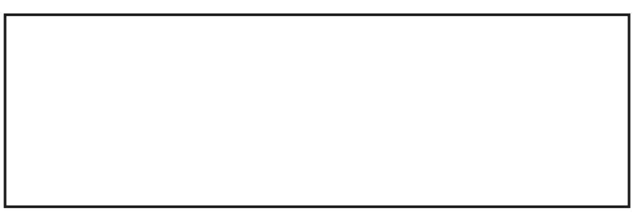
Find the perimeters in units and areas in square units of the following rectangles, if each unit is __ , and each square unit is □ .

1. 

2. 

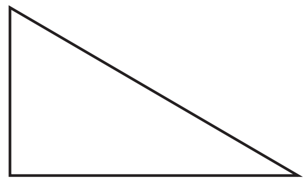
3. 

4. 

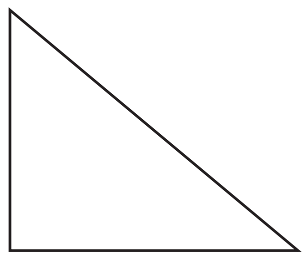
5. 

Find the perimeters in units and areas in square units of the following triangles, if each unit is —, and each square unit is □.

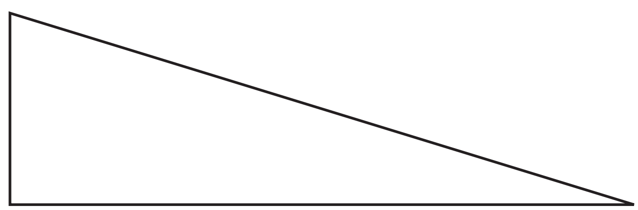
6. (Hint: Look at problem #1)



7. (Hint: Look at problem #4)



8. (Hint: Look at problem #5)



9. Describe, using pictures and/or words and/or math symbols, how you can calculate the area of a rectangle.

10. Describe, using pictures and/or words and/or math symbols, how you can calculate the area of a triangle.

Exercise 6-4:

Complete the times tables below.

×	2	3		11
4				
		15		
8			56	
12				

×		4	6	
5				
			36	
	18			135
10	20			

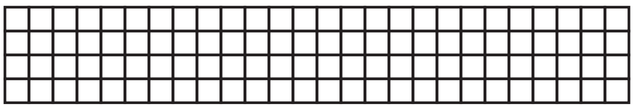
×	20	50	60	
40				
		3500		
	1600			7200
100				

6.2 The Distributive Property

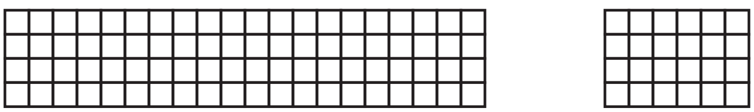
Vocabulary: The numbers being multiplied together in a multiplication problem are called the **factors**, and the answer to the multiplication problem is called the **product**.

We have seen how to multiply two numbers together up to $12 \times 12 = 144$. We've also seen that to estimate the product of two numbers that are bigger, we just round each number to one significant digit, then use our knowledge of multiplication by powers of 10. For example, we estimate the product $32,429.8 \times 192,487.103$ as $30,000 \times 200,000 = 6,000,000,000$.

The next step involves finding a way to get an exact product of numbers with two or more digits. We need the power of the **Distributive Property**. Let's say you want to find the product 4×26. We can see the four rows of 26 squares each in the following picture:



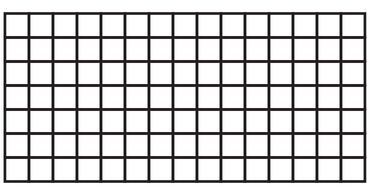
The ditributive property just says that we can take 4×26 and think of it as 4×20 plus 4×6 as seen in the following picture:



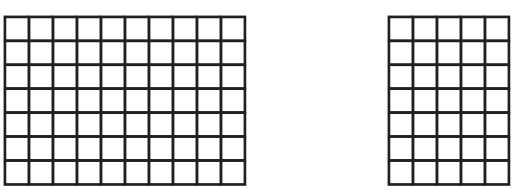
$4 \times 20 \quad + \quad 4 \times 6$

So, $4 \times 26 = 4 \times 20 + 4 \times 6 = 80 + 24 = 104$.

Here's another one. Let's say you want to find the product 7×15. We can see the seven rows of 15 squares each in the following picture:



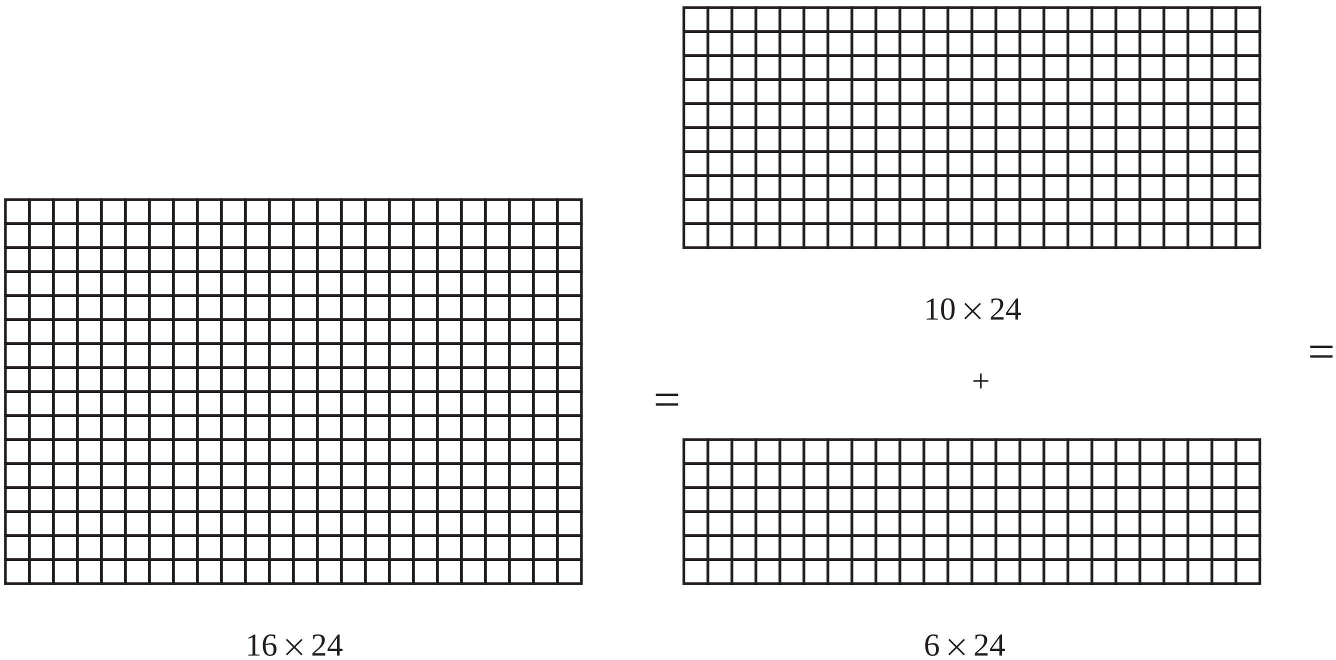
The distributive property just says that we can take 7×15 and think of it as 7×10 plus 7×5 as seen in the following picture:



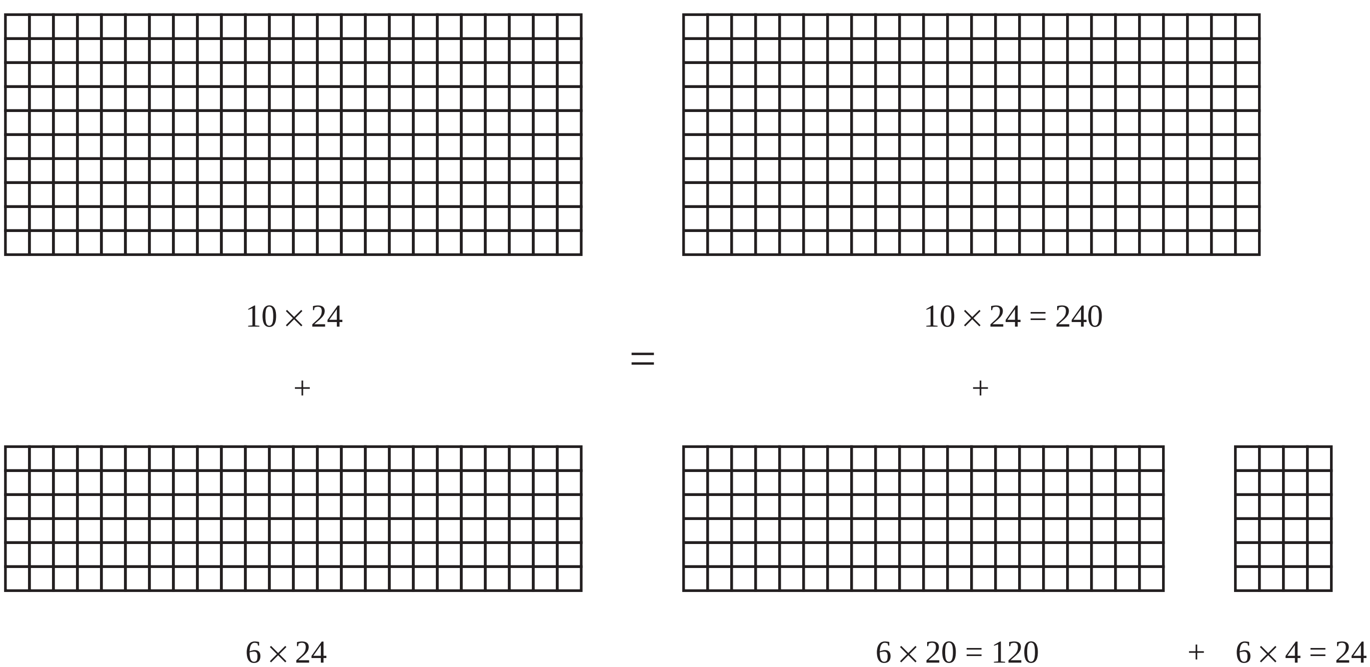
$7 \times 10 \quad + \quad 7 \times 5$

So, $7 \times 15 = 7 \times 10 + 7 \times 5 = 70 + 35 = 105$.

Let's try a harder one. 16×24 needs two steps of distributive property. First, we see 16×24 as 10×24 plus 6×24 as seen in the following picture:



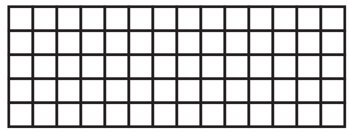
We already know that $10 \times 24 = 240$, so we just have to break up 6×24 into 6×20 plus 6×4.



So, $16 \times 24 = 10 \times 24 + 6 \times 20 + 6 \times 4 = 240 + 120 + 24 = 384$.

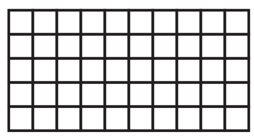
Exercise 6-5: For the following pictoral representations of the distributive property, fill in the missing factors to make the products reflect the pictures.

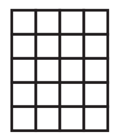
1.



=

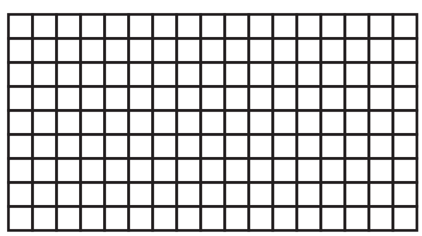
_____ × _____





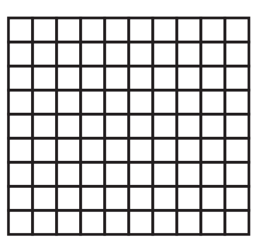
_____ × _____ + _____ × _____

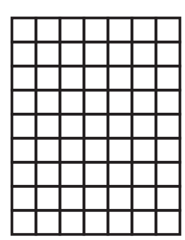
2.



=

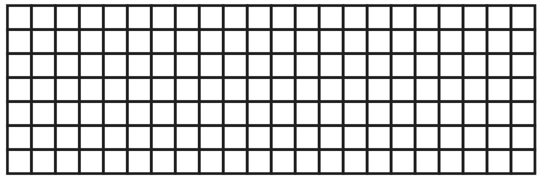
_____ × _____





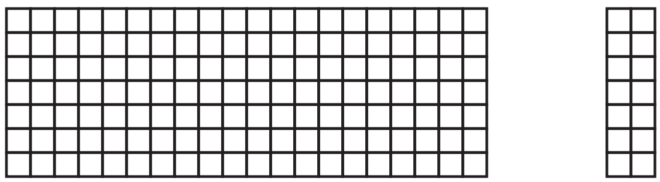
_____ × _____ + _____ × _____

3.



=

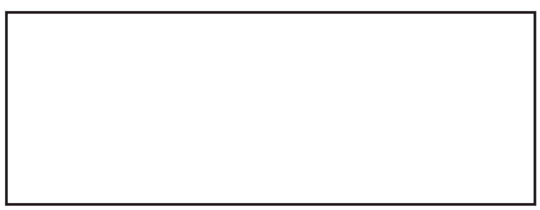
____ × ____



____ × ____ + ____ × ____

4.

63

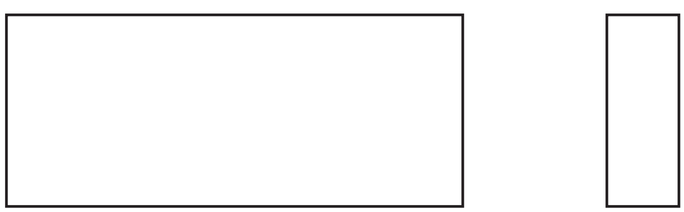


8

=

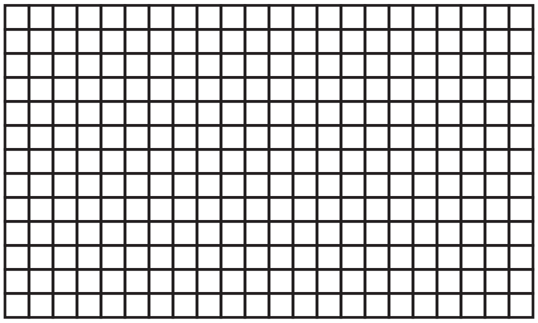
____ × ____

3



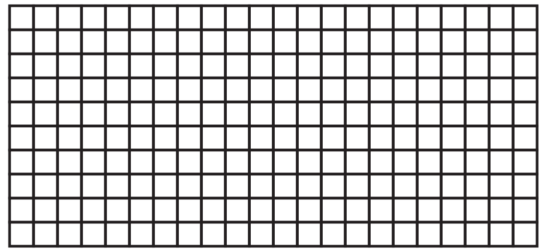
____ × ____ + ____ × ____

5.



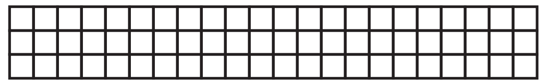
_____ × _____

=



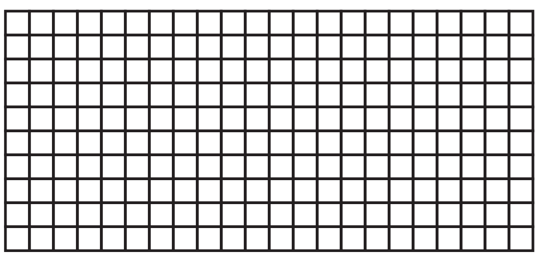
_____ × _____

+



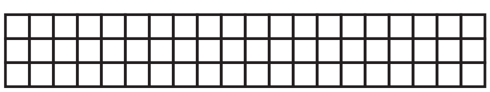
_____ × _____

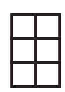
=



_____ × _____

+





_____ × _____ + _____ × _____

Exercise 6-6: In the previous exercise, you used pictures and distributive property to break up a difficult product into the sum of simpler products. In this exercise, write the math that is illustrated by the pictures in the problems from exercise 6-5 to find the value of the original products. The first one is done for you:

1. $5 \times 14 = 5 \times 10 + 5 \times 4 = 50 + 20 = 70$

2.

3.

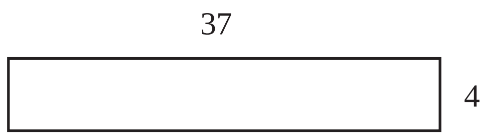
4.

5.

6.

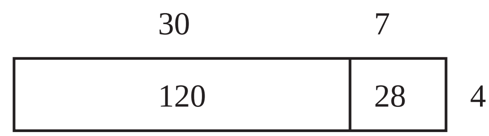
Exercise 6-7: Use distributive property to find the products.

Example 1:



4×37

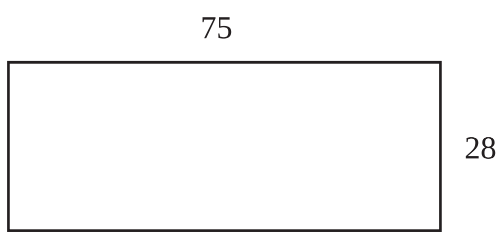
Solution:



$4 \times 37 = 4 \times 30 + 4 \times 7$

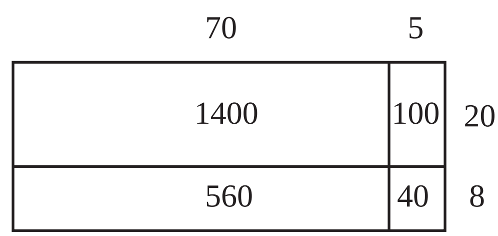
$= 120 + 28 = 148$

Example 2:



28×75

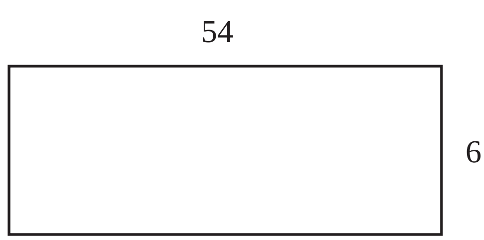
Solution:



$28 \times 75 = 20 \times 70 + 20 \times 5 + 8 \times 70 + 8 \times 5$

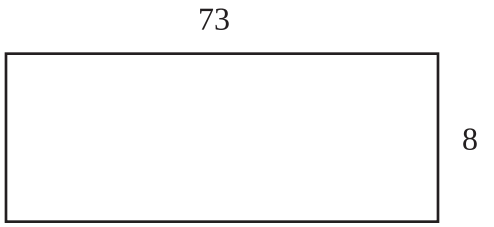
$= 1400 + 100 + 560 + 40 = 2100$

1.



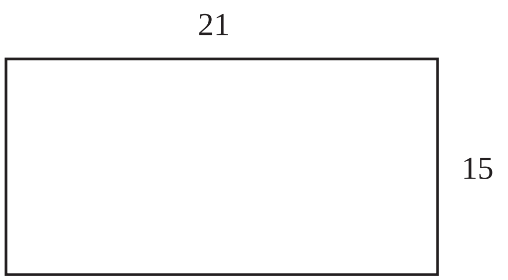
6×54

2.



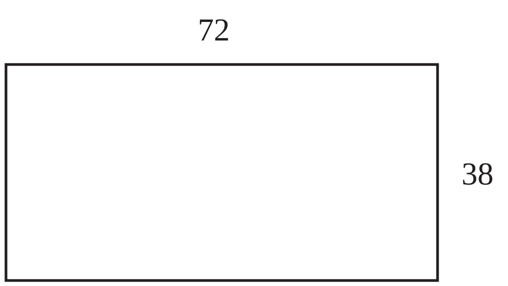
8×73

3.



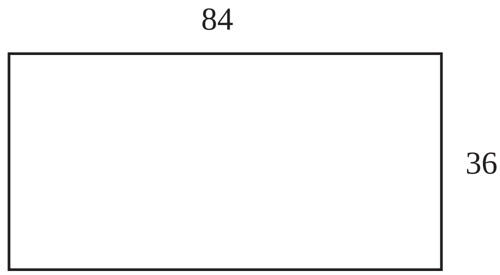
15×21

4.



38×72

5.



36×84

6. Multiply the numbers below by drawing a rectangle and breaking it into pieces like in the previous problems

(a) 44×27 (b) 18×57 (c) 29×72

Exercise 6-8: Using distributive property, re-write the following products as the sum of simpler products that you can figure out in your head. Then, add them together, to find the answer to the original product. You may use pictures to help if you like, but you are not required to.

1. 75×8

2. 52×14

3. 107×32

4. 530×61

5. 93×87

6. Compare and contrast (discuss what is similar and what is different) the distributive property method of multiplication with the standard "vertical method" of multiplication that you know from our past. Use the product 38×74 as the example to use in you discussion.

Find More Digits

Fill in the blanks to complete the multiplication problems below.

1.

$$\begin{array}{rrr} & 2 & \square \\ \times & & 8 \\ \hline 2 & \square & 4 \end{array}$$

2.

$$\begin{array}{rrr} & \square & 4 \\ \times & 2 & 7 \\ \hline 2 & \square & 8 \\ \square & 8 & \\ \hline \square & 1 & \square \end{array}$$

3.

$$\begin{array}{rrrr} & & 5 & 6 \\ & \times & 3 & \square \\ \hline & \square & 9 & \square \\ 1 & 6 & \square & \\ \hline \square & 0 & \square & 2 \end{array}$$

Exercise 6-9:

1. Estimate the following answers by first rounding the numbers to one significant digit.

   (a) How far do you travel if you drive for 6 hours at 62 mph?

   (b) How much will you earn if you are paid \$12.45 per hour for 38 hours?

   (c) How much will you pay for a year of service if you pay \$76 per month for your cell phone?

2. Repeat number 1 parts (a) and (c), but calculate the answers exactly this time.

3. How much carpet should Duane buy to cover a floor that is 12 feet 8 inches wide by 13 feet 4 inches long? Explain your reasoning.

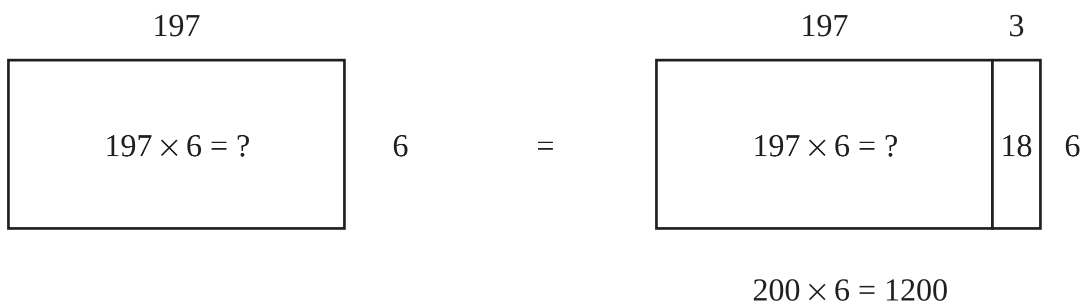
4. Maria is building a fence using boards that are $7\frac{5}{8}''$ wide. If she has 55 boards, give a reasonable over–estimate and a resonable under–estimate of the longest fence she can build.

Exercise 6-10: Tricky Math Problems using Distributive Property

Distributive Property can be used to make a product into a sum, like we've seen, or a difference.

Example: Conchi is buying new swivel chairs for her boutique business. She has to buy chairs for all 6 stations, and each chair costs \$197. How much will all of the chairs cost?

Solution: We can visualize this as the difference of two areas as follows:



So, $197 \times 6 = 200 \times 6 - 3 \times 6 = 1200 - 18 = 1182$, so, the chairs will cost \$1182.

For the following, first estimate the answer by rounding the factors to one significant digit before multiplying, then use distributive property to find the exact answer.

1. Find the product: 895×8

2. Find the product: 3998×27

3. Find the product: 48×6 (Can be done with addition OR subtraction.)

4. Find the area of a plot of land in the shape of a rectangle that measures 94 meters by 28 meters.

5. Joey wanted to take his buddies out to lunch. The lunch special where they like to go costs \$6.93 for each lunch and includes tax and tip. To feed all of his friends, he needs to buy 6 lunches. How much will it all cost

him?

6.3 Traditional Process of Multiplication

In the following activity we will see that the traditional method of multiplication is the same as the distributive property, just without seeing all of the details. Recall the traditional method of multiplying 36×72:

$$\begin{array}{cccc} & & 1 & \\ & & 7 & 2 \\ & \times & 3 & 6 \\ \hline & 4 & 3 & 2 \\ 2 & 1 & 6 & \\ \hline 2 & 5 & 9 & 2 \end{array}$$

Each of the numbers written down are part of the following distributive property solution:

	70	2	
	$70 \times 30 = 2100$	60	30
	$70 \times 6 = 420$	12	6

$36 \times 72 = 420 + 12 + 2100 + 60 = 2592$

The traditional process “keeps track” of the place values of the one digit numbers so that when you multiply, if you put your answers in the right columns, the answer comes out correct.

6.4 Factoring:

Note: Is a square a rectangle?
In short, yes. The definition of a rectangle is a four sided shape with four angles that have the same measure. The definition of a square is a four sided shape where all four sides have the same measure and all four angles have the same measure.

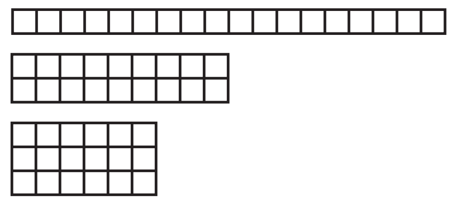
Activity
For a given area, there may be several rectangles with different dimensions that have that area.

> **Example:**
> For an area of 18 square units there are three different rectangles:

1×18:

2×9:

3×6:



> Notice that 6×3, 9×2, and 18×1 are the same as the three we already have, just sideways.

1. Repeat this example for rectangles with areas of 2 through 12 (there are eleven different areas, so 11 different problems). Use graph paper to help you sketch all of the different rectangles that have each given area.

2. Review your areas and their drawings and list all of the areas for which there was only one rectangle possible.

Factors

When two numbers are multiplied, the result is called their product and the two original numbers are called factors of their product. For example, the 3 and 5 in $3 \times 5 = 15$ are called factors of 15. The process of working backwards to split a number into the product of its factors is called factoring. You may have noticed in the previous activity that by finding the dimensions of a rectangle with a particular area, you were finding factors of that area. The factors of 18, for example, are exactly those numbers we used in the three rectangles we drew for the previous example. We say that 18 has six factors. Listed in pairs they are 1 and 18, 2 and 9, and 3 and 6. Numbers with only two factors, the number itself and 1, are called *prime* numbers. The first five primes are 2, 3, 5, 7, and 11. Please take a moment and think of where you have seen these numbers before. Notice that the number 1 is omitted from this list even though it fits the criteria. This is done by agreement among mathematicians because it makes things less complicated to leave 1 out.

Exercise 6-11:

1. List the first 10 prime numbers.

2. Notice that in the example for the previous activity, each of the rectangles has a different perimeter. Of the rectangles with area 18 square units, the one with a perimeter of 38 units has dimensions 1 by 18. Find the dimensions of the rectangle that satisfies each description below.

   (a) Area 14 square units, perimeter 30 units.

   (b) Area 20 square units, perimeter 24 units.

   (c) Area 24 square units, perimeter 20 units.

3. What are the dimensions of the rectangle with the smallest perimeter for an area of 36 square units? Include all of your reasoning in your solution.

4. Of all the rectangles with a perimeter of 20 units, what are the dimensions of the one with the smallest area? Include all of your reasoning in your solution.

5. Repeat the previous problem if you are allowed to use decimals.

Taxman

Work in pairs as a team to beat the Taxman. You will be given a number to work with for each game, such as 17. First write all the whole numbers from one through your number (in this case 1-17) like:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17

Then, make a table like this

Team	Taxman

with columns for the Taxman and the Team. To start the game, the team picks one of the numbers on the list, writes it in their column, and crosses it off the list. Next, they take all of the divisors of that number that have not already been crossed off, and write them in the taxman's column. They then cross those numbers off the list.

For example, let's say the team picks 10 first. The divisors of 10 are 1, 2, 5, and 10, but the team already has 10, so the Taxman gets 1, 2, and 5. The list and table after this round look like this:

~~1~~, ~~2~~, 3, 4, ~~5~~, 6, 7, 8, 9, ~~10~~, 11, 12, 13, 14, 15, 16, 17

Team	Taxman
10	1
	2
	5

At each round, the team picks a number still on the list, but there is one rule:

<u>The Rule:</u> Every number the team picks *must* have at least one divisor that has not been crossed off the list yet. That is, *the Taxman must get something on each round.*

For example, the team cannot now pick a 3, or a 6. (What other numbers can't be picked?) When the team can pick no more numbers, the Taxman gets all the numbers that are left. The numbers in each column are then added, to give the Team's score, and the Taxman's score.

6.5 Prime Factorization

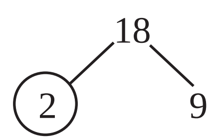
We have seen that *to factor* a whole number is to write it as a product of other numbers. For example, we can factor 12 as 3×4. This *factorization* is not unique; there are other ways to factor 12, such as 2×6. The *prime factorization* of a whole number is a way to factor a given number uniquely so that the factors are as simple as possible. For example, the prime factorization of 12 is $2 \times 2 \times 3$. It is a factorization of 12 since we have written 12 as a product of factors, and it is the prime factorization of 12 since 2 and 3, the factors, are each prime.

To find the prime factorization of a whole number:

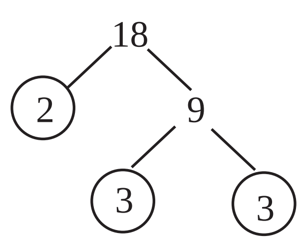
- Write the number at the top of your workspace.
- Think of *any* factorization of the number.
- Write the factors you thought of below the original number, then draw lines from the original number to each of the factors. Some people think these lines look like branches of a tree, so this process is called making a factor tree.
- Look at the numbers at the bottom of the lines. If the number is prime, you are done with that branch. Circle the prime factor. If the number is not prime, think of any factorization of that number, and write it below, connecting with lines.
- Continue until all the factors are prime.
- The prime factorization is the product of all the circled primes.

Example 1: Find the prime factorization of 18.

Solution: Start with any factorization. I thought of 2×9, but 3×6 works just as well. 1×18 doesn't work since it doesn't break 18 down any smaller and so doesn't get us closer to the result. Write the first branches of the tree:



Notice that the 2 is circled since it is prime. Continue to break down 9, the composite number, into 3×3. Circle the 3's since they are prime.



Since the ends of your branches are all prime numbers, the prime factorization is the product of all of the circled primes. therefore, the prime factorization of 18 is $2 \times 3 \times 3$.

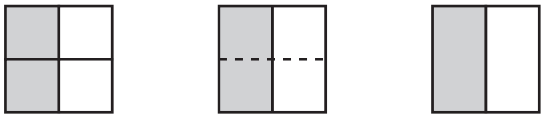
Exercise 6-12:

For each of the whole numbers between 2 and 25, write the word prime if the number is prime, and find the prime factorization of the number if the number is composite. The first few are done for you:

- 2 prime
- 3 prime
- $4 = 2 \times 2$
- 5 prime
- 6
- 7
- 8
- 9
- 10
- 11
- 12
- 13
- 14
- 15
- 16
- 17
- 18
- 19
- 20
- 21
- 22
- 23
- 24
- 25

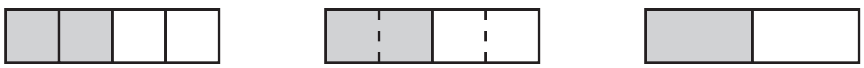
6.6 Equivalent Fractions Revisited

One way to look at finding a simpler, equivalent form of a fraction is to remove lines from the picture, making fewer, larger pieces. For example, by removing the middle horizontal line in the $\frac{2}{4}$ picture, it becomes $\frac{1}{2}$:



$\frac{2}{4}$ $\qquad$ $\frac{1}{2}$

If we are simplifying this way, it matters how we draw the fraction. We just drew fourths by making the four pieces in a 2×2 grid. If we had chosen a 1×4 grid instead, we would have had to remove two lines to make our equivalent fraction:



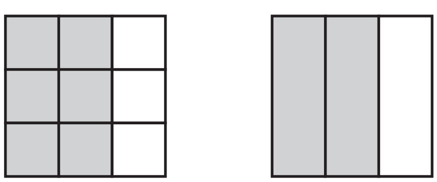
$\frac{2}{4}$ $\qquad$ $\frac{1}{2}$

Exercise 6-13:

- Draw a rectangle representing the given fraction.
- Draw another rectangle with the same amount shaded as the original picture, but with a line or some lines removed.
- Write the equivalent fraction.

Example: $\frac{6}{9}$

Solution:



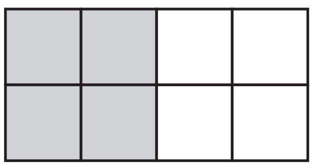
$\frac{6}{9}$ $\qquad$ $\frac{2}{3}$

1) $\frac{2}{6}$ 2) $\frac{2}{8}$ 3) $\frac{5}{10}$ 4) $\frac{6}{10}$ 5) $\frac{6}{8}$ 6) $\frac{8}{12}$ 7) $\frac{2}{12}$ 8) $\frac{3}{12}$

6.7 Factorization and Equivalent Fractions

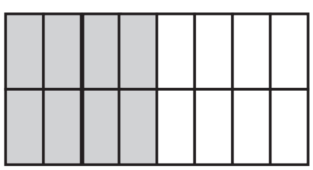
We saw in chapter 2 that fractions can be written in several *equivalent* ways. For example, we have seen that $\frac{4}{8}$ has many equivalent representations like $\frac{2}{4}$ and $\frac{8}{16}$. We have seen equivalence represented on a ruler (the fourth eighth inch mark is the same place as the second quarter inch mark etc.) and with picture representations like:

1 whole



$\frac{4}{8}$

$\frac{2}{4}$



$\frac{8}{16}$

In all of these, once the size of the whole is established, we can see that all of these fractions are equivalent to $\frac{1}{2}$.

Now that we understand factoring and factorization, we can investigate a pattern that occurs with equivalent fractions. Starting with $\frac{4}{8}$, we have the whole broken into 8 pieces, and 4 of them are shaded. If we remove two of the lines in the drawing, so that the whole is cut into only half as many pieces (denominator = 4), then the number of pieces shaded is also cut in half (numerator = 2). By looking at factored forms of the numerator and denominator of $\frac{4}{8}$, we can see a common factor of 2 which means we can easily cut each in half.

$$\frac{4}{8} = \frac{2\times 2}{2\times 4} = \frac{\textcircled{2}\times 2}{\textcircled{2}\times 4} = \frac{2}{4}$$

Similarly, if we start with $\frac{4}{8}$ and *double* the number of pieces in the whole which also doubles the number of pieces shaded, we see both the numerator and denominator of $\frac{4}{8}$ is doubled:

$$\frac{4}{8} = \frac{4 \times 2}{8 \times 2} = \frac{8}{16}$$

To get to simplest form, we start with $\frac{4}{8}$ and replace 4 and 8 with their prime factorizations. Then look for all common factors in the numerator and denominator, and remove them. We put 1 as a factor of 4, so that something is left in the numerator:

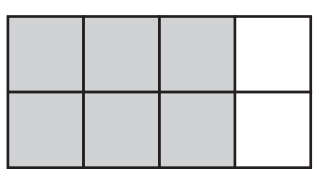
$$\frac{4}{8} = \frac{2 \times 2}{2 \times 2 \times 2} = \frac{1 \times \textcircled{2} \times \textcircled{2}}{2 \times \textcircled{2} \times \textcircled{2}} = \frac{1}{2}$$

Exercise 6-14: For the given fractions, do the following:

- Draw a picture representing the given fraction.
- Find the prime factorization of the numerator and denominator.
- Write an equation that is, "given fraction = fraction with numerator and denominator replaced with their prime factorizations".
- Circle all factors that are common in the numerator and denominator. You should have exactly the same number of factors circled in the top as you do in the bottom.
- Write the newly found simplest form of the original fraction.
- Draw a picture representing the simplest form.

Example: Given fraction is $\frac{6}{8}$

Solution:

The picture representation is:  $\frac{6}{8}$

The prime factorization of 6 is 2×3, and the prime factorization of 8 is $2 \times 2 \times 2$

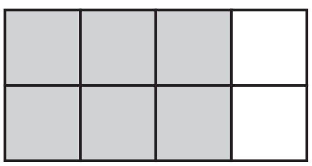
The equation looks like:

$$\frac{6}{8} = \frac{2\times 3}{2\times 2\times 2}$$

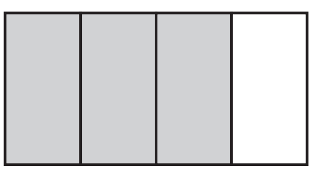
Circle the common 2 and then remove them to get:

$$\frac{6}{8} = \frac{\textcircled{2}\times 3}{\textcircled{2}\times 2\times 2} = \frac{3}{4}$$

The picture for $\frac{3}{4}$ lined up with the picture for $\frac{6}{8}$ so that the equivalence can be seen:



$\frac{6}{8}$



$\frac{3}{4}$

1. $\frac{2}{8}$

2. $\frac{3}{6}$

3. $\frac{4}{12}$

4. $\frac{6}{16}$

5. $\frac{3}{12}$

6. $\frac{8}{12}$

Multiples

The products created by multiplying a particular number by other numbers are called *multiples* of that number. The entries of the times table provide lists of multiples for the numbers at the beginning of that row or column. For example, the multiples of 4 are 4, 8, 12, 16, 20, and so on. Another way of saying this is that any number with 4 as a factor, is a multiple of 4.
A *common multiple* of two numbers is a number that both numbers are factors of (go into evenly). For example, 12 is a common multiple of 2 and 3 since both numbers are factors of 12.

A. Find two different numbers the given values share as a common multiple.

Example: 3 and 5
Solution: For the numbers 3 and 5 we could choose 15 and 30 as two different common multiples.
Checking a times table, the numbers 3 and 5 are both factors of 15, since $3 \times 5 = 15$ and 3 and 5 are both factors of 30,
since $3 \times 10 = 30$ and $5 \times 6 = 30$.

1. 2 and 5
2. 4 and 6
3. 4 and 8
4. 6 and 7
5. 9 and 12

B. Find the *smallest* multiple that each of the given values share. (Note: This smallest shared multiple is called the **Least Common Multiple** and is abbreviated **LCM**)

Example: 6 and 10
Solution: For 6 and 10 the smallest multiple is 30. While 60 ($= 6 \times 10$) is a multiple, it isn't the smallest.
If we check a times table for the multiples of 6 and 10, the first number they share in common is 30.

1. 2 and 3
2. 4 and 6
3. 4 and 8
4. 6 and 9
5. 9 and 12
6. 8 and 20
7. 12 and 18
8. 12 and 15
9. 12 and 20
10. 9 and 24

C. For each pair of numbers:

- Write the prime factorization of each number. Circle any prime factors that appear in each.
- Write the LCM of the two numbers.
- Write the prime factorization of the LCM.

1. 2 and 3
2. 4 and 6
3. 4 and 8
4. 6 and 9
5. 9 and 12

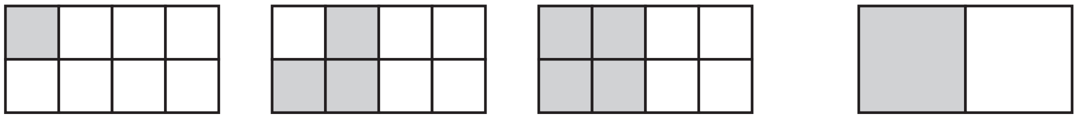
6. 8 and 20
7. 12 and 18
8. 12 and 15
9. 12 and 20
10. 9 and 24

D. There is a relationship between the prime factorizations of two numbers, and the prime factorization of their LCM. Looking at part C, describe any relationship that you can see. How can you use the prime factorizations of numbers to find their LCM?

E. Use the strategy that you came up with in part D to find the LCM of the following pairs of numbers.

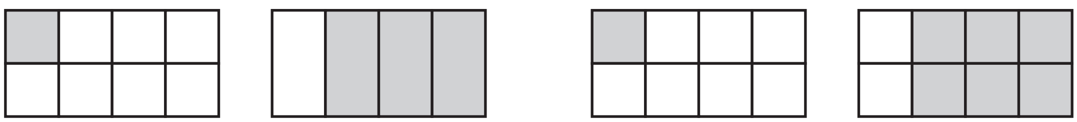
1. 10 and 15
2. 4 and 14
3. 6 and 8
4. 8 and 12
5. 15 and 20

We have seen that when adding or subtracting fractions, a common denominator is critical. For example, we have seen that $\frac{1}{8} + \frac{3}{8}$ is no problem since the denominators match...

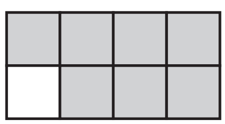


$$\frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1 \times 2 \times 2}{2 \times 2 \times 2} = \frac{1}{2}$$

but that $\frac{1}{8} + \frac{3}{4}$ needs some work before we can add because the denominators are different.



$$\frac{1}{8} + \frac{3}{4} = \frac{1}{8} + \frac{6}{8}$$



$$= \frac{7}{8}$$

Notice that to go from $\frac{3}{4}$ to $\frac{6}{8}$, we just add the middle line which doubles the number of pieces in the whole from 4 to 8, and doubles the number shaded from 3 to 6. One way to write this is: $\frac{3}{4} = \frac{3\times2}{4\times2} = \frac{6}{8}$. It's as if we introduce 2 as an extra factor in the numerator and 2 as an extra factor in the denominator.

If the pictures are simple enough, then we know how to make the denominators the same. If not, we need a systematic way to make equivalent fractions with matching denominators. It turns out that the easiest matching denominator to work with is the LCM of the given denominators!

The Golden Rule

A given fraction can be written as an equivalent fraction in two ways: remove a factor that appears in both the numerator and denominator, or introduce an extra factor in the numerator and denominator. In either case, whenever the operation is multiplication because we are taking out or putting in *factors*, we follow the Golden Rule of fractions: "Do to the top the same thing that you do to the bottom."

Let's see how this works with $\frac{1}{8} + \frac{3}{4}$:

- **Step 1:** Find the LCM of the given denominators.

  From exercise B number 3, you found that the LCM of 4 and 8 is 8.

- **Step 2:** Find a separate extra factor for each denominator to multiply times the original denominator to obtain the LCM.

  8 is already the LCM, so we don't need to change it at all. Leave $\frac{1}{8}$ as $\frac{1}{8}$.

  We need to find an extra factor to multiply by 4 to get 8. Since 8 is a multiple of 4 (it had to be, since it's the LCM), we know this is possible. In this case, we know that $4 \times 2 = 8$. The extra factor that we need for the fraction $\frac{3}{4}$ is 2.

- **Step 3:** Use the Golden Rule. Introduce the extra factor to both the denominator AND the numerator.

  $\frac{3}{4} = \frac{3\times 2}{4\times 2}$

- **Step 4:** Multiply out the factorizations that you came up with.

  $\frac{3\times 2}{4\times 2} = \frac{6}{8}$

  Now, we can add the fractions: $\frac{1}{8} + \frac{3}{4} = \frac{1}{8} + \frac{6}{8} = \frac{7}{8}$.

- **Step 5:** The final step is too simplify the answer if possible.

  Since the numerator, 7 and the denominator, 8 have no factors in common (7 is prime!), $\frac{7}{8}$ is already in simplest form.

Exercise 6-15: Add or subtract as indicated.

Example: $\frac{3}{4} - \frac{1}{6}$

Solution:

- **Step 1:** The LCM of 4 and 6 is 12.
- **Step 2:** $4 \times 3 = 12$ and $6 \times 2 = 12$, so $\frac{3}{4}$ needs an extra factor of 3 in the numerator and denominator, and $\frac{1}{6}$ needs an extra factor of 2 in the numerator and denominator.
- **Step 3:** $\frac{3}{4} = \frac{3\times3}{4\times3}$ and $\frac{1}{6} = \frac{1\times2}{6\times2}$.
- **Step 4:** $\frac{3\times3}{4\times3} = \frac{9}{12}$ and $\frac{1\times2}{6\times2} = \frac{2}{12}$.

  Now, we can subtract the fractions: $\frac{3}{4} - \frac{1}{6} = \frac{9}{12} - \frac{2}{12} = \frac{7}{12}$.
- **Step 5:** Again, since 7 is prime, there are no common factors. $\frac{7}{12}$ is already in simplest form, and is the final answer.

1. $\frac{1}{4} + \frac{1}{6}$
2. $\frac{1}{2} - \frac{1}{3}$
3. $\frac{3}{4} - \frac{5}{8}$
4. $\frac{5}{9} + \frac{1}{6}$
5. $\frac{4}{9} + \frac{7}{12}$
6. $\frac{3}{10} - \frac{2}{15}$
7. $\frac{5}{8} + \frac{5}{6}$

8. On the golf course, the grass on the fairway is supposed to be $\frac{3}{8}''$ high. If the grass on the fairway at the bottom of Sharp Park is $\frac{3}{4}''$ high, how much needs to be cut off?

9. While tiling their kitchen floor, Joyce and Pedro chose a pattern that alternated tiles that were $\frac{3}{5}''$ wide with bigger tiles that were $2\frac{5}{8}''$ wide. How wide were the two tiles put together if they also had to put $\frac{1}{8}''$ of grout between them?

10. Ahmed wanted to frame the window in his home office with decorative boarder strips. If his window frame was $52\frac{5}{8}''$ by $32\frac{3}{4}''$, what is the total length of the strips that he needs to go all the way around the window?

7 Division of Whole Numbers and Decimals

Money Activity

The bank has a pile of bills in denominations $10,000, $1000, $100, $10, $1, dimes and pennies. There are three members in each group. When you pick your money, make sure that the number of bills in each denomination is less than ten at any given time.

1. Start with the bank having only $376. Member #1 is the banker. Split the bank's money equally among all members, including the banker.
   (a) Record how much money each member receives.
   (b) Record the remainder, that is, how much money is left that can't be split evenly.

2. Start with the bank having only $6791. Member #2 is the banker. Split the bank's money equally among all members, including the banker.
   (a) Record how much money each member receives.
   (b) Record the remainder if any.

3. Start with the bank having only $1260. Member #3 is the banker. Split the bank's money equally among all members, including the banker.
   (a) Record how much money each member receives.
   (b) Record the remainder if any.

4. Each member take $1000.
   (a) Each member take your money and split it evenly into 5 separate piles.
      i. Record the amount in each pile.
      ii. Record the remainder, if any.
   (b) Each member take your $1000 and split it evenly into 6 separate piles.
      i. Record the amount in each pile.
      ii. Record the remainder, if any.
   (c) Each member take your $1000 and split it evenly into 3 separate piles.
      i. Record the amount in each pile.
      ii. Record the remainder, if any.
   (d) By splitting the $1000 into 5, 6, or 3 equal piles, when did you have the most money in each pile?

5. Start with the bank having only $14,000. Member #1 is the banker.

   (a) From the $14,000, give member #2 $1,500. Record how much money the bank still has.

   (b) Continue to give member #2 money from the bank, $1,500 at a time until there's not enough to continue. Record in the following table how much money the bank has left after each time:

Number of times $ given to #2	1	2	3	4	5	6	7	8	9	10
$ left over										

   (c) How many times does 1,500 go into 14,000?

   (d) How are the operations of subtraction and division related?

7.1 The Process of Division

Just as subtraction reverses the process of addition, multiplication, which is repeated addition, is reversed by division, or repeated subtraction.

To divide 6 by 2, we write $6 \div 2$ and by repeated subtraction we get,

$$6 - 2 = 4$$

$$4 - 2 = 2$$

$$2 - 2 = 0$$

3 two's

Which means we subtract 2 three times from 6 before nothing is left.

For larger numbers, like $448 \div 7$ we can subtract large amounts by multiplying first:

$$
\begin{aligned}
448 \div 7 &= 448 - \overbrace{7 - 7 - \cdots - 7}^{60\times 7=420} = 28 && \text{60 sevens} \\
&= 28 - \overbrace{7 - 7 - 7 - 7}^{4\times 7=28} = 0 && \underline{\text{4 sevens}} \\
& && \text{64 sevens}
\end{aligned}
$$

The operation of division is symbolized in a variety of ways. The symbol, $\div$, reminds us of a fraction where the dots represent the numbers in the fraction. In fact, $12 \div 4$, $12/4$ and $\frac{12}{4}$ are equivalent ways of representing twelve divided by four. Another common representation for division is the long division symbol, $\overline{)\quad}$, where $4\overline{)12}$ also represents twelve divided by four.

Vocabulary:
- The operation is called **division**.
- The number being divided (the 12 in $12 \div 4$) is called the **dividend**.
- The number used to divide (the 4 in $12 \div 4$) is called the **divisor**.
- Both the entire expression (the $12 \div 4$) and the result of the operation (3), are called the **quotient**.

The algorithm or method we typically see for long division is adapted from the idea of repeated subtraction above.

Example 1: To divide $7\overline{)448}$ we begin by subtracting the largest number of sevens we can calculate easily. Using place value, since 7×100 is too large ($700 > 448$), we choose the largest tens place number whose product with 7 is less than 440. Since $7 \times 60 = 420$ is the closest, we write a 6 above the tens place of the dividend and then subtract ($7 \times 60 =$) 420 from the dividend.

$$
\begin{array}{r}
6 \\
7\overline{)448}
\end{array}
\quad\longrightarrow\quad
\begin{array}{r}
6 \\
7\overline{)448} \\
\underline{420} \\
28
\end{array}
\quad\longrightarrow\quad
\begin{array}{r}
64 \\
7\overline{)448} \\
\underline{420} \\
28
\end{array}
\quad\longrightarrow\quad
\begin{array}{r}
64 \\
7\overline{)448} \\
\underline{420} \\
28 \\
\underline{28} \\
0
\end{array}
$$

We then start the process over again. Since we chose the largest tens place value to multiply by the first time, we move to the largest ones place now. Since $7 \times 4 = 28$ we choose 4 and write it in the ones place above the dividend. We then subtract 28 from 28 and since nothing remains we conclude that $448 \div 7 = 64$.

Notice that we can use multiplication to check that if we have the number 7, 64 times, we will get 448. Notice also, that if we use the distributive property to perform the multiplication, we get $7 \times 64 = 7 \times 60 + 7 \times 4 = 420 + 28$. These two terms in the distributive property sum are exactly the numbers that appeared in our division problem!

It is often the case that a quotient leaves a *remainder* or an amount left over that is not evenly divisible by the divisor. If four roomates want to split up 21 dishes, we have $21 \div 4 = 5$ with one plate left over. Since one plate isn't of much use in pieces, there's no way to share the remaining plate and it is left out as a remainder. Some things can be divided into fractional peices, however, as we will saw in the opening money activity. Consider the example below.

Example 2: Four roomates want to split up the \$623 security deposit they receive when their moving out of their apartment. In order to divide $4\overline{)623}$ we begin as before:

$$\begin{array}{r} 1 \\ 4\,\overline{)623} \\ 400 \\ \hline 223 \end{array} \quad\longrightarrow\quad \begin{array}{r} 15 \\ 4\,\overline{)623} \\ 400 \\ \hline 223 \\ 200 \\ \hline 23 \end{array} \quad\longrightarrow\quad \begin{array}{r} 154 \\ 4\,\overline{)623} \\ 400 \\ \hline 223 \\ 210 \\ \hline 23 \\ 20 \\ \hline 3 \end{array}$$

Now there are \$3 remaining to split between the four people. If we change out the dollars for dimes we have 30 dimes. This is accomplished in writing by placing a decimal point after the 4 in the top line and recognizing that we are now using tenths of a dollar so \$3 is equivalent to 30 tenths of a dollar, or 30 dimes:

$$\longrightarrow\quad \begin{array}{r} 154. \\ 4\,\overline{)623} \\ 400 \\ \hline 223 \\ 200 \\ \hline 23 \\ 20 \\ \hline 30 \end{array} \quad\longrightarrow\quad \begin{array}{r} 154.7 \\ 4\,\overline{)623} \\ 400 \\ \hline 223 \\ 200 \\ \hline 23 \\ 20 \\ \hline 30 \\ 28 \\ \hline 2 \end{array} \quad\longrightarrow\quad \begin{array}{r} 154.7 \\ 4\,\overline{)623} \\ 400 \\ \hline 223 \\ 200 \\ \hline 23 \\ 20 \\ \hline 30 \\ 28 \\ \hline 20 \end{array} \quad\longrightarrow\quad \begin{array}{r} 154.75 \\ 4\,\overline{)623} \\ 400 \\ \hline 223 \\ 200 \\ \hline 23 \\ 20 \\ \hline 30 \\ 28 \\ \hline 20 \\ 20 \\ \hline 0 \end{array}$$

From the thirty dimes, each person receives 7, or seven dimes, leaving 2 dimes remaining. If we change out the dimes for pennies, we now have 20 cents or twenty hundredths of a dollar. We accomplish this simply by placing a zero after the 2 and working from the hundredths place in the number in the top line. As a result we see each roomate receives \$154.75.

1. Do the problems below by repeated subtraction.

(a) $12\,\overline{)72}$ (b) $8\,\overline{)576}$

2. For each exercise below first perform the division and then write a word problem to go with it.

(a) $4\overline{)108}$

(b) $8\overline{)156}$

3. *Estimate* the following quotients by first rounding the numbers to one significant digit.

(a) $7\overline{)203}$

(b) $13\overline{)874}$

4. *Estimate* the answers to these questions by first rounding to one significant digit.

(a) If 7 people want to share $2468 evenly, how much should each person get?

(b) If Ramon drives his car 438 miles in 8 hours, what was his average speed?

(c) If Jenny can drive her car 370 miles on 12 gallons of gas, what is her fuel efficiency (in mpg)?

(d) How many feet are there in 378 inches?

7.2 Divisibility

A number is **divisible** by another number if when you divide the first by the second, there is no remainder. In other words, multiples are divisible by their factors.

Examples:

- 6 is divisible by the numbers 1, 2, 3 and 6.
- 15 is divisible by the numbers 1, 3, 5 and 15.
- 24 is divisible by the numbers 1, 2, 3, 4, 6, 8, 12 and 24.

The idea of divisibility (a division concept) is closely linked with the idea of factors (a multiplication concept).

For certain problems, it is nice to know before dividing what numbers it is divisible by. In the following exercises you will explore divisiblity and come up with some *divisibility rules*.

1. Numbers Divisible by 2:

   (a) List eight 3-digit numbers that are divisible by 2.

   (b) How can we tell whether a given number is divisible by 2?

   (c) Use your divisibility rule to figure out which of the following numbers are divisible by 2:

   1,392   57,120   228,301   77,754   300,005   713,293

2. Numbers Divisible by 5:

   (a) List eight 3-digit numbers that are divisible by 5.

   (b) How can we tell whether a given number is divisible by 5?

   (c) Use your divisibility rule to figure out which of the following numbers are divisible by 5:

   1,392   57,120   228,301   77,754   300,005   713,293

3. Numbers Divisible by 3:

(a) List the first twelve numbers bigger than 40 that are divisible by 3 in the table below . The first two are done for you.

(b) In the second column of the table, write the sum of the digits of the number written in the first column. The first two are done for you.

(c) In the thrid column, write "Yes" if the sum of the digits is divisible by 3, write "No" if the sum of the digits is not divisible by 3.

Multiple of 3	Sum of its Digits	Is the sum divisible by 3? Yes or No?
42	$4+2=6$	Yes
45	$4+5=9$	Yes

(d) Find a 3-digit number that is NOT divisible by 3. Is the sum of its digits divisible by 3?

(e) How can we tell whether a given number is divisible by 3?

(f) Use your divisibility rule to figure out which of the following numbers are divisible by 3:

1,392   57,120   228,301   77,754   300,005   713,293

Exercise 7-1:

1. Make up three word problems to go with $6\overline{)56}$.
In each problem the remainder should have to be handled differently. Write the problems on a separate piece of paper.

2. Which of the numbers below are divisible by three?

(a) 51 (b) 91 (c) 147 (d) 3812 (e) 4017 (f) 6112 (g) 10,206

3. Make up four different three digit numbers that are all divisible by three:

4. Make up four different three digit numbers that are all divisible by three:

5. Show your steps in answering these questions.

(a) If you drive at 60 mph for 3 hours, how far will you travel?

(b) If Joe drives 300 miles in six hours, what was his average speed?

(c) If you have $91 to split between 7 people, how much each person get?

(d) If Nick earns $12 per hour, how much money will he make working for 27 hours?

(e) Dolores earns $874 one week, If she worked for 38 hours that week, what was her hourly pay?

(f) There are 12 inches in a foot.

(i) How many inches are there in 8 feet?

(ii) How many feet are there in 228 inches?

(g) Maria wants to build a fence that is 90 feet long. If fence boards are 4 inches wide, how many fence boards will she need?

7.3 Estimation Again

We have seen that with addition, subtraction, and multiplication, we can *estimate* (find a number that is reasonably close to the answer in a quick and easy way) by rounding the numbers to one significant digit before performing the operations.

Exercise 7-2: Estimate the answers to the following addition, subtraction, and multiplication problems by rounding the numbers to one significant digit before performing the operations:

1. Find the area of a parking lot that measures 113.7 feet by 75.1 feet.

2. Find the perimeter of the same parking lot.

3. Julia's car blinks "low fuel" when she has used 12.38 gallons of her full tank. If she gets gas when she first sees the light come on, and gas is $2.79 per gallon, how much will it cost to fill her tank?

4. The techs in the car factory conducted an "almost" crash test for their new car. During the test, the driver started braking when the car was 83.85 feet from the wall and came to a complete stop when the car was 14.3 feet from the wall. How long was the skid?

5. Paulina needs 60 units to graduate. During her first semester, she took 12.5 units, but didn't pass a 3-unit class, so didn't get credit for it. The next semester, she took 9 units, and passed all of her classes. The following year, she passed all of her classes and took 13.5 units in the fall, and 17 units in the spring. How many more units does she need to have enough to graduate?

6. (Do this one exactly, not with an estimate...) How many square inches are in one square foot?

7. How many square inches are in 32.8 square feet?

Division estimation is more tricky. If the dividend (the number being divided up) is not a multiple of the divisor (the number going into the dividend), then there will be a remainder. This doesn't always get better when we round to one significant digit. Look at the following example:

Example 1: Estimate the quotient: $3,257 \div 7$

Solution: When we round to one significant digit, this becomes $3000 \div 7$ which isn't much easier since 7 doesn't go into 3 evenly (3 isn't big enough), and 7 doesn't go into 30 either, since the multiples of 7 are: $7 = 1 \times 7$, $14 = 2 \times 7$, $21 = 3 \times 7$, $28 = 4 \times 7$, $35 = 5 \times 7$, $42 = 6 \times 7$, etc. Since rounding to one significant digit doesn't make things easy enough to be worth it, another strategy is to:

- Round the divisor to one significant digit.
- Round the dividend to the nearest multiple of the rounded divisor that has at most two significant digits.
- Perform the division.

In this example, the divisor is 7 which is already only one digit. The dividend is 3,257. If we try to round to only one significant digit (the thousands place), 3,257 rounds to 3,000 which is not easy to see as a multiple of 7. Staying in the thousands place, the nearest multiple of 7 is 7000 which is too far away to be reasonable. If we round to two significant digits instead, the multiples of 7 are 1400, 2100, 2800, 3500, 4200, etc. The closest one to 3257 is 3500, so we will use that.

The division is now easy to perform since $35 \div 7 = 5$, so $350 \div 7 = 50$ and the for our estimate $3500 \div 7 = 500$. We can say the quotient $3,257 \div 7$ is approximately equal to 500.

Example 2: Estimate the quotient $54,728.2 \div 75$.

Solution: First round 75 to 80. Next, we look at 54,728.2 thinking of two-digit multiples of 8 to use for our 2 significant digits. Two-digit multiples of 8 near 54 or 55 are 40, 48, 56, 64, and 72. The closest is 56, so we will round 54,728.2 to 56,000.

Now, to perform the division, we know that $8 \times 7 = 56$, so $80 \times 700 = 56,000$. Therefore, $56,000 \div 80 = 700$ and $54,728.2 \div 75$ is approximately equal to 700.

Exercise 7-3: Estimate the following quotients using the strategy outlined on the previous page, then, using your calculator, find the actual quotients rounded to the nearest whole number:

1. $4,352 \div 6$

2. $53,722.9 \div 9$

3. $27,523 \div 37$

4. $337,495.75 \div 823.35$

5. $463,995 \div 61.99$

6. Make up a problem whose dividend has at least 3 significant digits and whose quotient has an estimate of 50.

Exercise 7-4: Suppose you are trying to collect information to help you choose a new car to buy. You start by looking only at costs. The car is going to be used primarily for commuting, so you are looking for a small, low cost automobile that gets good gas milage. You do some calculations and realize that you drive for your commute a total of 46.8 miles every work day. You also see that you average 20 miles of driving each weekend. Finally, you get two weeks of vacation every year, but plan to take a vacation in Hawaii every year, and will take a cab to the airport. It's the time of year when the car companies are offering no interest loans, as long as you make monthly payments and pay off the car in five years. Assume that gas prices stay at $3 per gallon. (Ha!!)

Use the information to complete the tables, the first with estimates, the second with actuals. Use the estimation strategies that you have learned for the estimates, and use your calculator to find the actuals rounded to the nearest whole dollar:

Car Type	List Price (in dollars)	Highway mpg	Annual Cost of Car Payment (estimate)	Annual Cost of Gas (estimate)	Total Annual Cost (estimate)
Honda Civic	18,400	40			
Saturn Ion	14,725	32			
Chevy Cobalt	16,150	32			
Hyundai Tiburon	18,745	28			
Jaguar Coupe	82,330	23			

Car Type	List Price (in dollars)	Highway mpg	Annual Cost of Car Payment (actual)	Annual Cost of Gas (actual)	Total Annual Cost (actual)
Honda Civic	18,400	40			
Saturn Ion	14,725	32			
Chevy Cobalt	16,150	32			
Hyundai Tiburon	18,745	28			
Jaguar Coupe	82,330	23			

7.4 Multiplying Fractions

Exercise 7-5:

1. (a) How many is $\frac{1}{2}$ of 8 apples?

   (b) If you divided 8 apples evenly into two groups, how many apples would you have in each group?

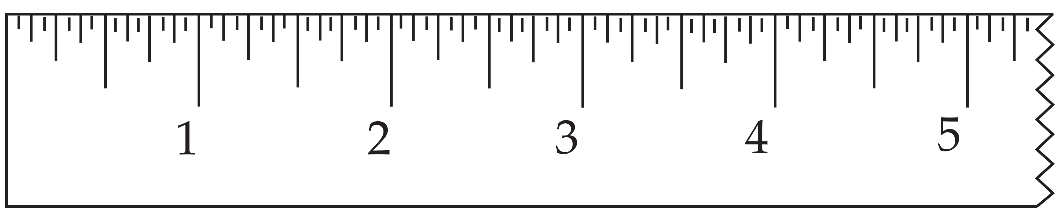
2. (a) How much is $\frac{1}{5}$ of \$10?

   (b) If you divided \$10 evenly into 5 groups, how much money would you have in each group?

3. (a) 12 people showed up to the church fundraiser. $\frac{1}{3}$ of them were men. How many men showed up to the fundraiser?

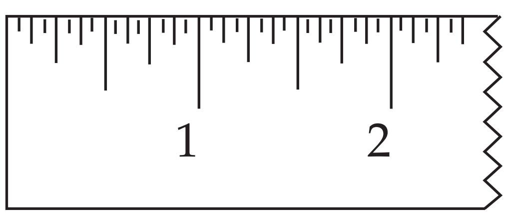
   (b) At the church fundraiser, they broke the 12 people up into 3 groups. How many were in each group?

4. (a) How far is $\frac{1}{2}$ of 4 inches?



   (b) If you divided 4 inches evenly into two pieces, how many inches would you have in each piece?

5. (a) How far is $\frac{1}{4}$ of 2 inches?



   (b) If you divided 2 inches evenly into four pieces, how many inches would you have in each piece?

6. What operation (addition, subtraction, multiplication, or division) did you use to calculate part (b) in problems 1-5?

7. If eight friends each brought $\frac{1}{2}$ of a pizza to the party, how many pizzas would they have all together?

   (a) Justify the solution using repeated addition and pictures.

   (b) Since repeated addition of the same value is equivalent to multiplication, how could you use multiplication to get the same answer as in part (a)?

8. If you had $\frac{1}{5}$ ten times, how much would you have?

   (a) Write the solution using repeated addition.

   (b) Write the solution using multiplication.

9. If you had $\frac{1}{3}$ twelve times, how much would you have?

   (a) Write the solution using repeated addition.

   (b) Write the solution using multiplication.

10. If you had $\frac{1}{4}$ two times, how much would you have?

    (a) Write the solution using repeated addition.

    (b) Write the solution using multiplication.

Exercise 7-6: For the following products:

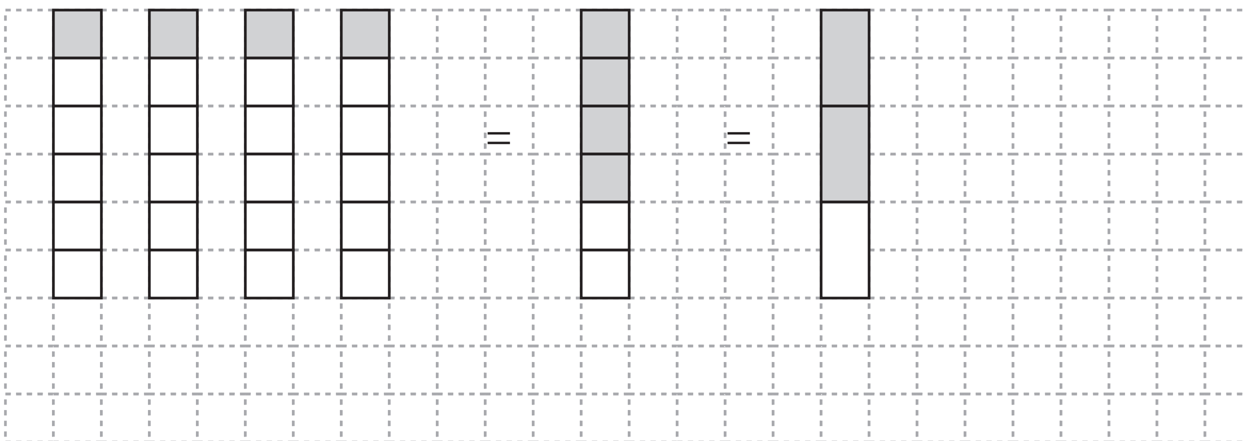
- Write the solution using repeated addition. (Write fraction answers in simplest form.)
- Draw a picture representing the repeated addition.
- Write the solution using multiplication. (Write fraction answers in simplest form.)

Example: $4 \times \frac{1}{6}$

Solution:

Using repeated addition: $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2\times 2}{2\times 3} = \frac{2}{3}$

The picture:



Using multiplication: $4 \times \frac{1}{6} = \frac{4\times 1}{6} = \frac{2\times 2}{2\times 3} = \frac{2}{3}$

1. $2 \times \frac{3}{4}$

2. $6 \times \frac{2}{3}$

3. $4 \times \frac{3}{8}$

4. $3 \times \frac{5}{6}$

5. We have seen that $7 \times \frac{3}{4} = \frac{3}{4} + \frac{3}{4} + \ldots + \frac{3}{4} = \frac{21}{4}$ or $5 + \frac{1}{4} = 5\frac{1}{4}$. Explain in words (or pictures if you like) how to multiply in order to get this same result. Explain why this works.

Exercise 7-7: For the following products:

- Write the solution using repeated addition. (Write fraction answers in simplest form.)
- Write the solution using multiplication. (Write fraction answers in simplest form.)

Example: $3 \times \frac{5}{6}$

Solution:

Using repeated addition: $\frac{5}{6} + \frac{5}{6} + \frac{5}{6} = \frac{15}{6} = \frac{3\times5}{3\times2} = \frac{5}{2}$ or $2 + \frac{1}{2} = 2\frac{1}{2}$.

Using multiplication: $\frac{3\times5}{6} = \frac{3\times5}{3\times2} = \frac{5}{2}$ or $2 + \frac{1}{2} = 2\frac{1}{2}$.

1. $2 \times \frac{3}{8}$

2. $6 \times \frac{1}{4}$

3. $5 \times \frac{3}{10}$

4. $4 \times \frac{5}{6}$

5. $8 \times \frac{3}{4}$

6. $2 \times \frac{1}{6}$

Exercise 7-8: Let's look at another way of looking at a fraction times a whole number:

1. What number is $\frac{1}{2}$ of 10?

   (a) Write the solution using division. (Write fraction answers in simplest form.)

   (b) Write the solution using multiplication. (Write fraction answers in simplest form.)

2. What number is $\frac{1}{4}$ of 12?

   (a) Write the solution using division. (Write fraction answers in simplest form.)

   (b) Write the solution using multiplication. (Write fraction answers in simplest form.)

3. What number is $\frac{1}{5}$ of 10?

   (a) Write the solution using division. (Write fraction answers in simplest form.)

   (b) Write the solution using multiplication. (Write fraction answers in simplest form.)

4. What number is $\frac{1}{4}$ of 28?

   (a) Write the solution using division. (Write fraction answers in simplest form.)

   (b) Write the solution using multiplication. (Write fraction answers in simplest form.)

5. What number is $\frac{1}{6}$ of 18?

   (a) Write the solution using division. (Write fraction answers in simplest form.)

   (b) Write the solution using multiplication. (Write fraction answers in simplest form.)

6. What number is $\frac{1}{2} \times 6$?

   (a) Write the solution using division. (Write fraction answers in simplest form.)

   (b) Write the solution using multiplication. (Write fraction answers in simplest form.)

7. What number is $\frac{1}{3} \times 6$?

   (a) Write the solution using division. (Write fraction answers in simplest form.)

   (b) Write the solution using multiplication. (Write fraction answers in simplest form.)

8. What English word means the same as "$\times$" in the examples above?

Group Activity: Looking at the previous few pages, you can see a pattern for how multiplication of a fraction times a whole number or multiplication of a whole number times a fraction works. In your groups, write a procedure for how to multiply a whole number A times a fraction $\frac{b}{c}$, as in $A \times \frac{b}{c}$. When you are done, make up 3 different problems like this that are different from the problems on the previous pages.

Exercise 7-8: Show all your work as you complete the following:

1. A family's budget is \$3,600 per month. $\frac{1}{2}$ goes towards rent, $\frac{1}{9}$ towards food, $\frac{1}{4}$ towards bills, $\frac{1}{60}$ towards entertainment, and the rest to savings. How much money does the family pay in each category? What fraction goes towards savings?

2. A rectangle measures 4 inches long by $\frac{5}{8}$ inches wide.

   (a) Estimate the perimeter by rounding each measurement to the nearest whole inch before computing.

   (b) Find the actual perimeter.

   (c) Estimate the area by rounding each measurement to the nearest whole inch before computing.

   (d) Find the actual area.

3. After finishing the Holiday shopping, $\frac{2}{5}$ of her holiday savings account was left. If she had $1245 saved originally, how much was left after the shopping?

4. Working at the golf course, Martin found 135 golf balls in the lake. $\frac{2}{3}$ of them were too soggy to use and had to be thrown away. How many golf balls were too soggy?

5. On a fishing trip, the family caught 24 fish. $\frac{5}{8}$ of them were too small, and had to be thrown back. How many fish did they keep?

6. Joan drank a 12 oz. bottle of Burpo and Carol drank a 16 oz. can of Carbon Light. "I drank $\frac{1}{3}$ more than you did," said Carol. I drank $\frac{1}{4}$ less," said Joan. Who is right and why ?

7. A student wrote the following on his arithmetic test: $\frac{1\not{6}}{\not{6}4} = \frac{1}{4}$ and $\frac{2\not{6}}{\not{6}5} = \frac{2}{5}$. He claims he has discovered that sixes always cancel. His teacher thinks this was just an accident. Who is right? If the student is, explain to his teacher why his method always works. If his teacher is correct, find an example for the student where his method fails.

8. A box of Sugar Glops© cereal contains $\frac{7}{10}$ of a pound of cereal and a box of Astro Puffs contains 11 ounces of cereal. (1 pound = 16 ounces). They have he same food value (none) and the same taste (sweet) and cost the same. Which is the better buy?

9. A jet plane cruises at 450 mph. for $3\,\frac{2}{5}$ hours. How far does it travel?

10. After the party, only half of the birthday cake was left. The next morning, I ate $\frac{2}{5}$ of what was left. What fraction of the original cake did I eat?

11. You are planning to do some baking for an office party. You figure out you will have enough cookies for the party and enough to bring to your mother-in-law's house on Sunday if you make 6 batches. The amount of each ingredient needed for each batch is given below:

Amount	Ingredient
$2\frac{1}{2}$ squares	unsweetened chocolate
$\frac{1}{2}$ cup	butter
2 cups	flour
$\frac{1}{2}$ teaspoon	baking soda
1 teaspoon	baking powder
$\frac{1}{4}$ teaspoon	salt
$1\frac{1}{4}$ cups	white sugar
1 teaspoon	vanilla extract
$\frac{2}{3}$ cups	sour cream
2 cups	semi-sweet chocolate chips

Fill in the following table with the amounts needed of each ingredient to make 6 batches:

Amount	Ingredient
	unsweetened chocolate
	butter
	flour
	baking soda
	baking powder
	salt
	white sugar
	vanilla extract
	sour cream
	semi-sweet chocolate chips

12. You are planting bushes in your back yard. Each bush needs $\frac{3}{4}$ pounds of fertilizer and $4\frac{3}{8}$ feet of flexi-board to frame the hole. How much fertilizer and how much flexi-board do you need to buy to have enough materials for 12 bushes?

7.5 Fraction $\times$ Fraction

We have seen that a fractional part of a whole number can be obtained through multiplication, that is, that $\frac{1}{2}$ **of** 6 is the same as $\frac{1}{2} \times 6$, and that $A \times \frac{b}{c} = \frac{b}{c} \times A = \frac{A \times b}{c}$. The final step is to look for common factors in the numerator and denominator, so that you can write your result in simplest form.

To visualize what the fractional part of a fraction means, it is very important to be clear about the size of "the whole" in your problem.

Exercise 7-9: Find the following products by:

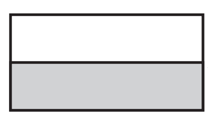
- Rewrite the product with the word "of" replacing $\times$.
- Draw a picture of the size of the "whole" you will be using for your other pictures.
- Draw a picture representing the second fraction in the product.
- Draw a picture in which the first fraction of the second fraction is shaded.
- Write the answer to the product in lowest terms.

Example 1: What is $\frac{1}{2} \times \frac{1}{2}$?

Solution: $\frac{1}{2} \times \frac{1}{2}$ is equivalent to $\frac{1}{2}$ of $\frac{1}{2}$.
Suppose your whole is


,

so that one-half shaded would look like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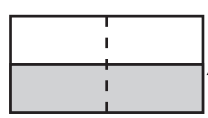
To shade only half of one-half, we can cut the rectangle in half like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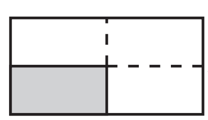
then only shade half of the previously shaded part like


.

From this picture we can see that $\frac{1}{2}$ of $\frac{1}{2}$ is $\frac{1}{4}$, or using a math equation, $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

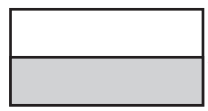
Example 2: What is $\frac{2}{3} \times \frac{1}{2}$?

Solution: $\frac{2}{3} \times \frac{1}{2}$ is equivalent to $\frac{2}{3}$ of $\frac{1}{2}$.

Again, we will start with one whole as


,

so that one-half shaded would look like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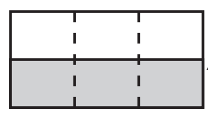
This time, we need to cut everything in thirds like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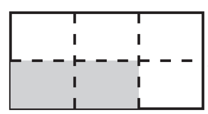
so that we can shade $\frac{2}{3}$ of the previously shaded part: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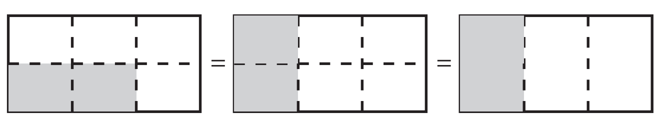
From this picture we can see that $\frac{2}{3}$ of $\frac{1}{2}$ is $\frac{2}{6}$, but since $6 = 2 \times 3$, we know that $\frac{2}{3} \times \frac{1}{2} = \frac{2}{6} = \frac{1 \times 2}{3 \times 2} = \frac{1}{3}$. Notice that if we move one of the shaded sixths in the picture to the first column, we can see how $\frac{2}{6} = \frac{1}{3}$.



1. $\frac{3}{4} \times \frac{1}{3}$

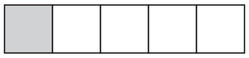
2. $\frac{2}{3} \times \frac{3}{8}$

3. $\frac{4}{5} \times \frac{5}{8}$

4. $\frac{5}{2} \times \frac{1}{2}$

Multiplying Fractions

We've seen that $20 \times \frac{1}{5}$ means $\frac{1}{5}$ added 20 times and this gives us $\frac{20}{5} = \frac{4 \times 5}{5} = 4$. How should we interpret $\frac{1}{5} \times 20$? One way of looking at this is to recognize that multiplication is *commutative*, meaning that 3×4 and 4×3 give the same result. Therefore, $\frac{1}{5} \times 20$ should be the same as $20 \times \frac{1}{5} = \frac{20}{5}$.
This doesn't really explain what $\frac{1}{5} \times 20$ means, however. Remember that $\frac{1}{5}$ is represented as one shaded box out of five:



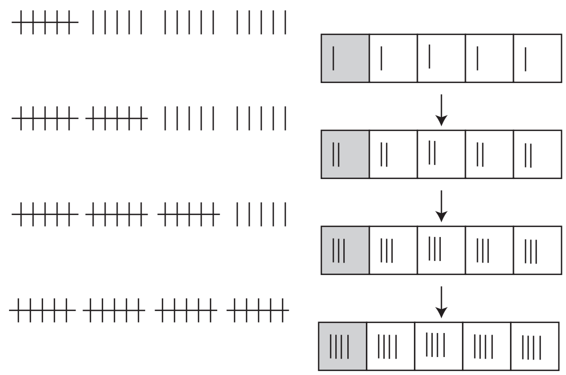
$\frac{1}{5} \times 20$ or one fifth of twenty, means that we have twenty things distributed equally into five slots and we collect one of them. To see this, think of twenty toothpicks placed into five boxes one at a time:

The first five are placed in the boxes,

Then the next five are placed in the boxes,

Then the third group of five are placed in the boxes,

Finally, the last five are placed in the boxes.

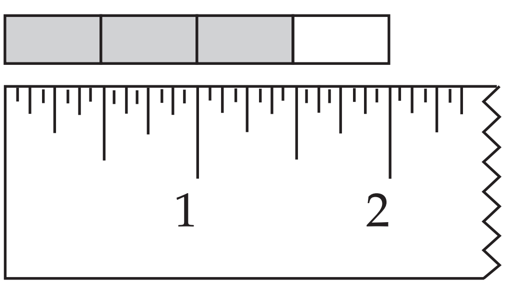


When we count up the toothpicks in the shaded box, there are 4, so $\frac{1}{5} \times 20 = 4$.

In the same way, if we multiplied $\frac{2}{5} \times 20$ we would have the same rectangle but with two shaded boxes:

So the result is 8. Notice that this is consistent with our expectations that $\frac{2}{5} \times 20 = 20 \times \frac{2}{5} = \frac{40}{5}$.
Exercise: Make a similar drawing to find $\frac{3}{4} \times 20$.

Another way to see how the denominator divides things into groups is through length. We can think of the product $\frac{3}{4} \times 2$ as taking three quarters of two inches. So we divide two inches into quarters and count up three of them:

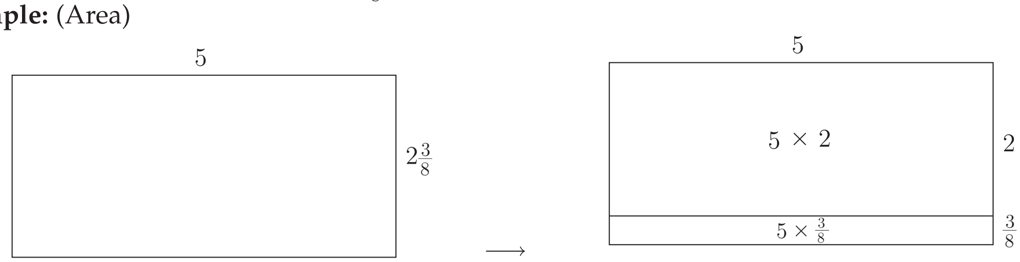


From this it's pretty easy to see that $\frac{1}{4}$ of $2''$ is a half inch, so $\frac{3}{4} \times 2 = 1\frac{1}{2}$.

Multiplying Mixed Fractions
When we multiply mixed fractions, it's good to start like we have previously, by using repeated addition.
Example: (Repeated Addition)
Multiply $5 \times 2\frac{3}{8}$ by repeated addition.

$$\begin{aligned} 5 \times 2\frac{3}{8} &= 2\frac{3}{8} + 2\frac{3}{8} + 2\frac{3}{8} + 2\frac{3}{8} + 2\frac{3}{8} \\ &= \underbrace{2+2+2+2+2}_{10} + \underbrace{\frac{3}{8} + \frac{3}{8} + \frac{3}{8} + \frac{3}{8} + \frac{3}{8}}_{\frac{15}{8}} \\ &= 10 + 1\frac{7}{8} \\ &= 11\frac{7}{8} \end{aligned}$$

We can also represent multiplication as we did previously, using an area model. Since $5 \times 2\frac{3}{8}$ would give us the area of a rectangle with sides 5 and $2\frac{3}{8}$, we get the following:
Example: (Area)



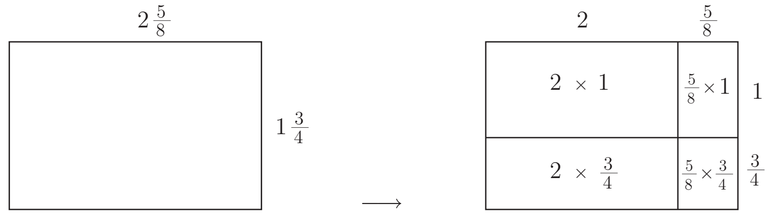
Which again gives us $10 + \frac{15}{8} = 11\frac{7}{8}$.

A third possibility is to multiply $5 \times 2\frac{3}{8}$ by rewriting $\frac{3}{8}$ as an improper fraction.

Example: (Improper Fraction)
Since $2\frac{3}{8} = \frac{8}{8} + \frac{8}{8} + \frac{3}{8} = \frac{17}{8}$, we can use the process of multiplication we developed previously to get:
$5 \times 2\frac{3}{8} = 5 \times \frac{17}{8} = \frac{5 \times 17}{8} = \frac{85}{8}$

$$\begin{aligned} 5 \times 2\frac{3}{8} &= 5 \times \frac{19}{8} \\ &= \frac{5 \times 19}{8} \\ &= \frac{95}{8} \\ &= 11\frac{7}{8} \end{aligned}$$

Using the same reasoning, in order to multiply $2\frac{5}{8} \times 1\frac{3}{4}$ we can look at it as though we're finding the area of a rectangle with sides, $2\frac{5}{8}$ and $1\frac{3}{4}$. Then much as we did with distribution in the previous sections, we can break up the rectangle into smaller pieces.



Then the area or the product is: $2\frac{5}{8} \times 1\frac{3}{4} = 2 \times 1 + 2 \times \frac{3}{4} + \frac{5}{8} \times 1 + \frac{5}{8} \times \frac{3}{4}$

$$= 2 + \frac{6}{4} + \frac{5}{8} + \frac{15}{32}$$
$$= 2 + \frac{48}{32} + \frac{20}{32} + \frac{15}{32}$$
$$= 2 + \frac{83}{32}$$
$$= 4\frac{19}{32}.$$

Exercises: Multiply like in the example above.

1. $5 \times 3\frac{5}{8}$ (b) $2\frac{3}{4} \times 10$ (c) $4\frac{1}{2} \times 2\frac{7}{8}$

Exercises:

1. Miriam is cutting blocks for the wall she is building. If she needs 20 blocks that are $14\frac{3}{8}''$ each, how much wood does she need?

2. Ralph ate $\frac{1}{3}$ of an apple pie. Later Alice ate $\frac{3}{4}$ of the remainder. What part of the total pie did Alice eat?

3. Find $2\frac{2}{5} \times 3\frac{3}{4}$ using a picture.

4. Henry spends $\frac{3}{14}$ of his monthly income for rent and $\frac{2}{11}$ of what is left on food. If he has \$540 left, what is his monthly income?

5. How many inches are there in each fraction of a foot:

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{3}$

(d) $\frac{1}{6}$ (e) $\frac{2}{3}$ (f) $\frac{3}{4}$

6. Find $\frac{1}{2}$ of $4\frac{1}{4}''$.

7. Find $\frac{1}{2}$ of $5\frac{3}{8}''$.

Dividing Fractions

In this section we will look at dividing fractions. We already have a sense for this from money. We know that when we divide a dollar into quarters, we get four of them. Mathematically this means $1 \div \frac{1}{4} = 4$. Think about the operations that might explain this result.

In order to understand division with fractions better, it will help to have a visual aid. Complete the questions below.

1. Use the tangram puzzle shown below to find the fraction of the whole square represented by each shape.

A: ____________

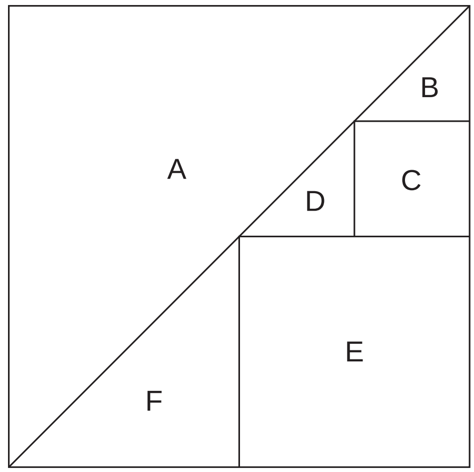
B: ____________

C: ____________

D: ____________

E: ____________

F: ____________



2. Use the figure in (1) to answer these questions.

(a) What fraction of the square is A? __________ , what fraction of the square is F? __________

(b) How many times does F go into A? __________

(c) Write the question in part (b) as a division problem using the fractions you wrote for A and F.

3. Use the figure in (1) to answer these questions.

(a) What fraction of the square is C? __________ , What fraction of the square is E? __________

(b) How many times does C go into E? __________

(c) Write the question in part (b) as a division problem using the fractions you wrote for C and E.

4. Use the figure in (1) to answer these questions.

(a) What fraction of the square is A? __________ , What fraction of the square is C? __________

(b) How many times does C go into A? __________

(c) Write the question in part (b) as a division problem using the fractions you wrote for A and C.

From your answers to questions 1–4, you should have a sense for the way fractions work with division. You saw that the smaller fractions of the square go into the larger ones a whole number of times. Now let's try to improve our understanding with more examples.

Remember that division is just repeated subtraction, as we saw with dividing whole numbers.

Example: (Repeated Subtraction)

$3 \div \frac{1}{2}$

Another way to say this is, "How many times can $\frac{1}{2}$ be subtracted from 3?"

$3 - \frac{1}{2} = 2\frac{1}{2} \Longrightarrow$ subtracted 1 time

$2\frac{1}{2} - \frac{1}{2} = 2 \Longrightarrow$ subtracted 2 times

$2 - \frac{1}{2} = 1\frac{1}{2} \Longrightarrow$ subtracted 3 times

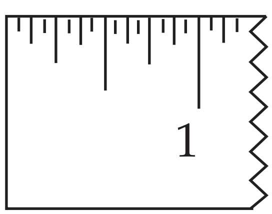
$1\frac{1}{2} - \frac{1}{2} = 1 \Longrightarrow$ subtracted 4 times

$1 - \frac{1}{2} = \frac{1}{2} \Longrightarrow$ subtracted 5 times

$\frac{1}{2} - \frac{1}{2} = 0 \Longrightarrow$ subtracted a total of 6 times

Therefore, $3 \div \frac{1}{2} = 6$

We can also see this nicely using a ruler. Suppose we want to divide $\frac{1}{2}'' \div \frac{1}{16}''$. Using the ruler below we can count the number of sixteenths that fit into $\frac{1}{2}''$. So $\frac{1}{2}'' \div \frac{1}{16}'' = 8$.



Exercise:
Use the ruler above to help you answer $\frac{3}{4} \div \frac{3}{16}$.

Another way we use to understand division with fractions is to compare them through common denominators. Then the pieces are the same size so we can concentrate on the number of pieces.

Example: (Common Denominators)

Find $\frac{3}{4} \div \frac{1}{8}$ by first writing the fractions with the same denominators.

Since $\frac{3}{4} = \frac{6}{8}$ we can rewrite this problem as $\frac{6}{8} \div \frac{1}{8}$.

But this just means how many times does $\frac{1}{8}$ go into $\frac{6}{8}$?

We can see that the answer is 6 but it follows directly from dividing the numerators: $6 \div 1 = 6$.

We can apply this logic to more challenging fractions.

Example: (Common Denominators)

Find $\frac{4}{5} \div \frac{3}{8}$ by first writing the fractions with the same denominators.

The LCM of 5 and 8 is 40 so we rewrite the fractions,

$\frac{4}{5} \longrightarrow \frac{32}{40}$ and $\frac{3}{8} \longrightarrow \frac{15}{40}$

So now we have $\frac{32}{40} \div \frac{15}{40}$.

Since the pieces (the denominators) are the same size, we can ignore them and divide the number of pieces (the numerators): $32 \div 15$ which is a little more than two times.

We can also write the answer as a fraction: $\frac{32}{15}$ which should give us some insight into the arithmetic of dividing fractions. We'll return to this question in a minute but first try the following exercises using the common denominator method.

Exercises

(a) $\frac{1}{2} \div \frac{1}{8}$ (b) $\frac{5}{8} \div \frac{5}{16}$ (c) $\frac{7}{8} \div \frac{2}{3}$

Now let's return to the result from the previous example. We concluded that $\frac{4}{5} \div \frac{3}{8} = \frac{32}{15}$. Notice that the numerator of the answer comes from 4×8 while the denominator comes from 5×3. It certainly appears as though we multiplied $\frac{4}{5}$ by $\frac{8}{3}$.

Let's see if we can explain why this last conclusion would make sense mathematically.

Remember the Golden Rule, that fractions are equivalent as long as we multiply the top number by the same value we multiply the bottom number by. Also remember that any number divided by 1 is the same as the original number, e.g. $\frac{7}{1} = 7$.

Now we are ready to see why $\frac{4}{5} \div \frac{3}{8}$ should equal $\frac{4}{5} \times \frac{8}{3}$. We know that $\frac{4}{5} \div \frac{3}{8}$ is equivalent to writing $\frac{\frac{4}{5}}{\frac{3}{8}}$. If we apply the Golden Rule and multiply by $\frac{8}{3}$, we get

$$\frac{\frac{4}{5} \quad \times\frac{8}{3} \longrightarrow}{\frac{3}{8} \quad \times\frac{8}{3} \longrightarrow} \quad \frac{\frac{4}{5} \times \frac{8}{3}}{1} = \frac{4}{5} \times \frac{8}{3}$$

And this shows why dividing fractions is the same as flipping the second fraction and multiplying.

Exercises

1. Explain like you were teaching a student new to fractions why we "invert and multiply," to calculate $\frac{2}{5} \div \frac{1}{3}$.

2. A girl spends $\frac{1}{3}$ of her savings and loses $\frac{2}{3}$ of the money remaining. She then has $12. How much did she start with?

3. What is the area, in square miles, of a farm $1\frac{3}{10}$ miles long by $\frac{2}{3}$ mile wide ?

4. A length of cloth $6\frac{7}{8}$ yards long is divided into 5 equal pieces. How long is each piece ?

5. A box of modeling clay weighs $45\frac{1}{2}$ pounds. How many $1\frac{3}{4}$ lb. packages will one box fill ?

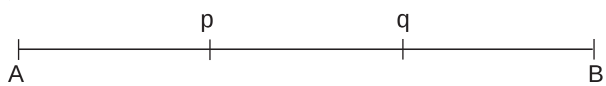
6. Maurice is making shelves for his apartment. He wants the shelves to be at least $2\frac{1}{2}$ feet long. How many shelves can he make from a piece of wood that is 8 feet long?

7. How long should Maurice make each shelf in #6 if he doesn't want any wood left over?

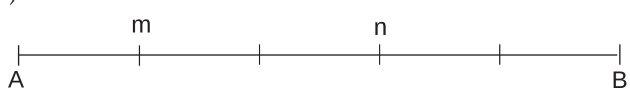
8. If the minimum width for a parking space is $8\frac{1}{2}$ feet, what is the maximum number of parking spaces you can fit perpendicular to a 100 foot long street (assume there are no driveways or tow away zones)?

9. Each line shows a trip from point A to point B. In each case identify the fraction of the trip travelled at the given letter.

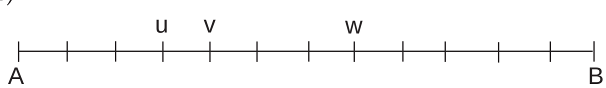
(a)



(b)

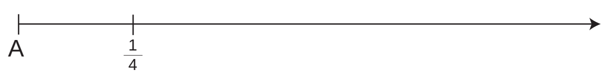


(c)

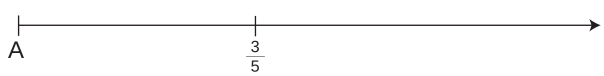


10. If B marks the end of the trip, show where B should be located on the line if the fraction identifies the fraction of the trip from A to B completed at that point.

(a)



(b)



11. Write the following ounce measurements as fractions of 1 pound. (Remember there are 16 ounces in 1 pound).

(a) 8 oz. (b) 10 oz. (c) 12 oz. (d) 6 oz.

Decimal Fractions

Decimal fractions are a special case of fractions that use powers of ten, e.g. 10, 100, 1000, etc., as denominators. These are exactly the numbers used in place value (see page 7) and as a result we extend the place value definition to give an alternative representation of the fraction. Just as the number to the left of the ones place is designated for tens (because ten ones add up to ten), the number to the right of the ones place is designated for ten*ths* (because ten tenths add up to one). We will write three tenths as $\frac{3}{10}$ or equivalently as 0.3.

Multiplication with decimals

A fraction is nothing more than a convenient way of dividing something up (the denominator) and counting up the number of pieces (the numerator). When we take $\frac{3}{4}$ of \$60, we divide sixty into four groups of \$15 each and then add up three of the groups to get \$45.

Example:

Find 0.4×60.
Because 0.4 is the same as writing $\frac{4}{10}$, multiplying 0.4×60 is the same as $\frac{4}{10} \times 60$ or taking $\frac{4}{10}$ of 60. As with other fractions, we can divide first, to get $60 \div 10 = 6$ and then add up four of the sixes to get 24.

Equivalently we can write this mathematically as $\frac{4}{10} \times 60 = \frac{4 \times 60}{10} = \frac{240}{10}$ and then divide to get 24.

Like the hundreds place, two spaces to the left of the ones, the decimal representation for hundred*ths* is two spaces to the right of the ones place. So 0.07 is equivalent to $\frac{7}{100}$. The decimal 0.27 is equivalent to 2 tenths and 7 hundredths so this is $\frac{2}{10} + \frac{7}{100}$ or $\frac{20}{100} + \frac{7}{100} = \frac{27}{100}$.

Exercises:

1. Write the following decimal fractions as fractions with appropriate denominators.

   (a) 0.5 (b) 0.05 (c) 0.009 (d) 0.34 (e) 0.85 (f) 1.7

2. Write the following fractions as decimal fractions.

   (a) $\frac{1}{10}$ (b) $\frac{3}{100}$ (c) $\frac{89}{100}$ (d) $\frac{127}{1000}$ (e) $\frac{237}{100}$

Multiplying decimals is technically no different than multiplying fractions. We will develop the familiar shortcuts but we will start with an example.

Example:

Find 0.32×20.

We'll start by converting 0.32 to $\frac{32}{100}$.

$$\text{Then } \frac{32}{100} \times 20 = \frac{32 \times 20}{100} = \frac{640}{100} = \frac{64}{10} = 6\frac{4}{10}$$

Because $\frac{4}{10}$ has 10 in its denominator, it is equivalent to the decimal fraction 0.4 and it is preferable to write the result, $6\frac{4}{10}$, as 6.4.

Exercises:

1. Multiply these decimal fractions by first converting to fraction form. Write your answers in decimal form.
   (a) 0.3×20 (b) 0.08×15 (c) 12×0.2 (d) 0.24×42

2. Do you recognize a shorter way to get your results rather than converting to fractions and then back to decimals? Explain.

Example:

Find 0.6×0.45.
Again we start by converting 0.6 to $\frac{6}{10}$ and 0.45 to $\frac{45}{100}$.

$$\text{Then } \frac{6}{10} \times \frac{45}{100} = \frac{6 \times 45}{10 \times 100} = \frac{270}{1000} = \frac{27}{100}$$

Because $\frac{27}{100}$ has 100 in its denominator, it is equivalent to the decimal fraction 0.27 and it is preferable to write it this way.

Exercises:

1. Multiply these decimal fractions by first converting to fraction form. Write your answers in decimal form.

(a) 0.3×0.2 (b) 0.08×1.5 (c) 0.12×0.2 (d) 0.24×0.42

2. Compare your answers with those to the previous exercise set. Again, do you recognize a shorter way to get your results rather than converting to fractions and then back to decimals? Explain.

3. Multiply 0.4×0.35 using any shortcut you described above.

Using decimals with division

The process of division usually ends when the remainder is smaller than the divisor and we leave it either as a remainder or as a fraction. Using decimals gives us an alternative way of representing a fractional remainder as well as an alternative way of getting it.

Just as we did with dividing money between members of a group, whenever there isn't enough of a larger bill to go around, we change it out for ten of some smaller quantity. If there are four people and only three dollars, we don't stop dividing, we just exchange the dollars for dimes. If there aren't enough dimes to split up, we exchange the dimes for pennies.

The process of division with decimal fractions is exactly like splitting up money except we don't have to stop at pennies (hundredths) when we divide up an amount. If there were a unit of money equal to a tenth of a penny, we would be able to keep dividing, and so it is with decimals.

Example:

Divide $3 \div 8$.

This is the same as saying we want to divide $3 among 8 people and we can see that there aren't enough dollars to go around so our first step is to exchange our three dollars for 30 dimes.
Now $30 \div 8 = 3$ with 6 dimes left over, so everyone gets 3 dimes.

Since we can't divide up the 6 dimes, we exchange them for 60 pennies and again divide, $60 \div 8 = 7$ with four pennies remaining.

Now normally we would be done since there aren't any coins equivalent to a tenth of a penny, but let's imagine there are. We trade in our 4 pennies for 40 of these tenths of a penny, and now $40 \div 8 = 5$ so each person get's 5 tenths of a penny, or 5 thousandths of a dollar.

Totalling the coins each person gets, we have 3 dimes, 7 pennies, and 5 tenths of a penny or equivalently,

3 tenths of a dollar, 7 hundredths of a dollar, and 5 thousandths of a dollar. In decimal notation this looks like, 0.3 + 0.07 + 0.005 which adds up to 0.375.

Example:

Repeat the example above using the division algorithm.

$$\begin{array}{rl} & 0 \\ 8 & \overline{)3} \end{array}$$

Since 8 doesn't divide three evenly, we write a 0 above the ones place.

$$\begin{array}{rl} & 0. \\ 8 & \overline{)30} \end{array}$$

We exchange three ones for 30 tenths (dimes) and note this above by marking a decimal point (to show we're now in tenths).

$$\begin{array}{rl} & 0.3 \\ 8 & \overline{)30} \\ & \underline{24} \\ & \;\;6 \end{array}$$

8 divides 30 three times with a remainder of 6 tenths (meaning $8 \times 0.3 = 2.4$).

$$\begin{array}{rl} & 0.3 \\ 8 & \overline{)30} \\ & \underline{24} \\ & \;\;60 \end{array}$$

Again, we exchange 6 tenths for 60 hundredths (notice how the places line up).

$$\begin{array}{rl} & 0.37 \\ 8 & \overline{)30} \\ & \underline{24} \\ & \;\;60 \\ & \;\;\underline{56} \\ & \;\;\;\;4 \end{array}$$

8 divides 60 seven times with a remainder of 4 hundredths (meaning $8 \times 0.07 = 0.56$).

$$\begin{array}{rl} & 0.375 \\ 8 & \overline{)30} \\ & \underline{24} \\ & \;\;60 \\ & \;\;\underline{56} \\ & \;\;\;\;40 \\ & \;\;\;\;\underline{40} \\ & \;\;\;\;\;\;0 \end{array}$$

Finally, we exchange 4 hundredths for 40 thousandths and since $8 \times 0.005 = 0.04$ we are done. So $3 \div 8 = 0.375$.

Exercises:

1. Divide the following numbers using decimals.

(a) $1 \div 2$ (b) $1 \div 4$ (c) $3 \div 5$ (d) $\frac{1}{8}$ (e) $\frac{5}{8}$

2. Sometimes we get numbers with decimals that never end but repeat forever. This isn't a problem as long as we recognize the pattern and write the result with a line over the repeating part. Divide the fractions below and express the answer with a line over the repeating part.

(a) $1 \div 3$ (b) $2 \div 3$ (c) $1 \div 9$

3. The decimal expressions for numbers divided by 7 have an interesting pattern. See if you can find it by doing the following problems.

(a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$ (d) $\frac{4}{7}$

4. See if the pattern you found in #3 can help you predict the decimals for $\frac{5}{7}$ and $\frac{6}{7}$.

Division with Decimals

As with multiplication, we can use the same methods we developed for dividing fractions to divide decimal fractions.

Example:

Find $0.6 \div 0.02$.

$$\begin{aligned}\text{Writing as fractions we get } \frac{6}{10} \div \frac{2}{100} &= \frac{6}{10} \times \frac{100}{2} \\ &= \frac{600}{20} \\ &= 30\end{aligned}$$

Exercises:

1. Find these quotients by converting the decimals to fractions first.

(a) $0.8 \div 0.004$ (b) $0.12 \div 0.03$ (c) $1.5 \div 0.05$ (d) $0.42 \div 0.007$

The more familiar version of the division procedure for decimals is to "move" the decimals in both the divisor and the dividend the same number of times until the decimal no longer remains in the divisor.

This looks like

$0.04\overline{)0.72} \quad \longrightarrow \quad 4\overline{)72}$

But there's rarely much explanation to support it.

Too see what is happening, let's write the quotient as a fraction: $\frac{0.72}{0.04}$. Obviously it would be nice if we didn't have the decimals involved so we look at the divisor, 0.04, and since it is in hundredths, we use the Golden Rule and multiply both the top and the bottom of the fraction by 100:

$$\frac{0.72 \overset{\times 100}{\longrightarrow}}{0.04 \underset{\times 100}{\longrightarrow}} = \frac{72}{4} = 18$$

This is equivalent to the method of moving the decimal but provides some explanation for what's going on.

Exercises:

1. Complete these division problems by any method.

(a) $\frac{0.42}{0.03}$ (b) $\frac{6.6}{0.3}$ (c) $0.06\overline{)0.39}$ (d) $1.2\overline{)0.0\,36}$

Exercises:

1. Find the answers to these two problems. Compare the methods you used to get your answers.

(a)
$$\begin{array}{r} 1\;3\;7 \\ +\quad 9\;2 \\ \hline \end{array}$$

(b)
$$\begin{array}{r} .1\;3\;7 \\ +\;.0\;9\;2 \\ \hline \end{array}$$

2. Explain how to find the answer to this problem. Explain why your method gives the correct answer.

$$\begin{array}{l} \quad .1\;3 \\ \times\;.5 \\ \hline \end{array}$$

3. Holly and Kevin went to Canada on a vacation. Holly changed money before leaving. For every US$0.82 , she got one Canadian dollar. Kevin changed his money when he got there. For each US$1.00 he got $1.20 in Canadian money. Who got the better deal?

4. Convert 1.735 to a fraction and then explain a general rule for converting decimals to fractions.

5. James needed to divide a 12 foot 2×4 into 5 pieces of equal length. He used his calculator to find that each piece needed to be 2.4 feet long so he began cutting pieces that were 2 feet, 4 inches. When James was done, how long was his last remaining piece?

6. Check the appropriate box for the result of each operation. You should do these in your head.

(a) $5.65 + 9.47$	☐ Less than 15	☐ More than 15
(b) $21.06 - 15.49$	☐ Less than 6	☐ More than 6
(c) 4.08×9.12	☐ Less than 36	☐ More than 36
(d) 4.08×9.12	☐ Less than 37	☐ More than 37
(e) $17.8 \div 3.2$	☐ Less than 6	☐ More than 6

7. Complete these equivalent fractions.

(a) $\frac{4}{5} = \frac{\quad}{100}$ (b) $\frac{1}{5} = \frac{\quad}{100}$ (c) $\frac{3}{4} = \frac{\quad}{100}$ (d) $\frac{1}{2} = \frac{\quad}{100}$ (e) $\frac{2}{3} = \frac{\quad}{100}$

Order of Operations and Exponents

Discussion Questions:

1. At Santa Cruz Beach Boardwalk, I bought an All-Day Unlimited Rides Wristband for \$26.95, and 3 cotton candies for \$1.25 each.
   (a) What two different operations (addition, subtraction, multiplication, division) could be used to calculate the total cost for my day at The Boardwalk?
   (b) Write in words how to calculate the total cost, including what order you use the numbers given in the problem and operations from part (a).
2. Each month, the household spends \$450 for food and \$1950 for rent.
   (a) What two different operations (addition, subtraction, multiplication, division) could be used to calculate the total yearly budget for food and rent?
   (b) Write in words how to calculate the total yearly budget for food and rent, including what order you use the numbers given in the problem and operations from part (a).

Order of Operations Agreement

Whenever there is more than one operation used to solve a problem, it is important to know which order to calculate. If you calculate in the wrong order, you will get the wrong answer! Since the order makes a difference, mathematicians came to an agreement (the Order of Operations Agreement) about how to indicate what order the operations must be performed. For multiplication and addition the agreement is:

- Multiplication must be performed before addition, regardless of the order it is written.
- If the addition should be performed first, use grouping symbols (parenthesis).

Example: Evaluate $3 + 5 \times 2$.

Solution: The multiplication is performed first, so $3 + 5 \times 2 = 3 + 10 = 13$.

Example: How can we make 3 plus 5 times 2 equal to 16?

Solution: To indicate that the addition should be performed first, use grouping symbols: $(3 + 5) \times 2 = 8 \times 2 = 16$. (Note: When parenthesis are used, and a number is multiplied by the result in the parenthesis, the $\times$ symbol is optional. The previous expression could have been written: $(3 + 5)2$ or more commonly, $2(3 + 5)$.)

Exercises: Find the value of the following expressions:

1. $7 + 4 \times 3$

2. $3(5 + 11)$

3. $\frac{1}{2} \times \frac{4}{3} + 1\frac{2}{3}$

4. $13 + 1.25 \times 0.6$

Addition and Subtraction from Left to Right

Suppose you are keeping track of your checking account balance. The starting balance is $150.75. You then write a check for $120. Since your balance is getting low, you make a deposit of $40. The correct way to write the mathematical expression for this situation is also the most straightforward way, that is:

$150.75 - 120 + 40.$

To evaluate it, we do the addition and subtraction from left to right and get:

$150.75 - 120 + 40 = 30.75 + 40 = 70.75.$

Compare this to a similar situation in which you start the month with a balance of $2500, then write checks for $32.95, $150, and $535. In this case, you must use grouping symbols to indicate that you need to total the checks before subtracting from your balance. Inside the parenthesis, the addition can be done in any order you like:

$2500 - (32.95 + 150 + 535) = 2500 - (32.95 + 685) = 2500 - (717.95) = 1782.05.$

Exercises: Find the value of the following expressions:

1. $28 - 4 + 13$

2. $32 + 17 - 20$

3. $\frac{1}{2} + (1\frac{2}{3} - \frac{3}{4})$

4. $28 - 7 \times 3.5$

Multiplication and Division from Left to Right

Multiplication and division work the same as addition and subtraction. That is, if both occur in the same problem, the operations are done from left to right unless grouping symbols indicate otherwise.

Exercises: Find the value of the following expressions:

1. $72 \div 9 \times 5$

2. $14 \times 3 \div 7$

3. $42 \div (3 \times 2)$

4. $\frac{1}{2} \times (\frac{2}{3} \div \frac{5}{6})$

5. $0.28 \times 1.4 \div 7$

Exponents

Recall when we were finding prime factorizations, that sometimes a prime factor appeared more than once:

$24 = 8 \times 3 = 2 \times 4 \times 3 = 2 \times 2 \times 2 \times 3.$

Since the factor 2 is repeated, it would be nice if there was a short-hand way to write it. Luckily, just like multiplication can be used to indicate repeated addition of the same value, **exponents** are used to indicate repeated multiplication of the same value. In our example, instead of writing $8 = 2 \times 2 \times 2$, we write $8 = 2^3$ with the little 3 above and to the right of the 2 written to indicate that the 2 is multiplied by itself 3 times. It is important to write the exponent a little smaller than the 2 (which is called the **base**) so that it is clear to all reading it that it is an exponent. Using this notation, the prime factorization of 24 can be written as:

$24 = 2^3 \times 3$

Exercises: Write the following products using exponential notation:

1. $2 \times 2 \times 3 \times 5 \times 5$

2. $2 \times 3 \times 3 \times 7$

3. $2 \times 2 \times 2 \times 2 \times 5$

4. $3 \times 3 \times 5 \times 5 \times 5$

5. $3 \times 3 \times 7 \times 7 \times 7$

Exercises: Write the prime factorization of the following numbers. If a factor is repeated, use exponential notation.

1. 36

2. 400

3. 250

4. 2400

5. 625

Exercises: Find the value of the following exponential expressions.

1. 3^2

2. 2^5

3. $\left(\frac{2}{3}\right)^4$

4. $\left(\frac{1}{5}\right)^3$

5. $(0.012)^2$

Suppose you started a small business with $5,000. Then, during the first year, the value of the business doubled. In the second year, the value of the business doubled again. In the third year, after lots of hard work, miraculously, the value doubled again! To calculate how much the business is worth after the three years, we have to multiply 5,000 by 2 for each time the value doubled.

Value of the business after three years $= 5,000 \times 2 \times 2 \times 2 = 40,000$

or using exponential notation because of the repeated 2's:

$5,000 \times 2^3 = 40,000$, so that the business is worth $40,000 at the end of the 3 yrs.

Notice that even though the multiplication in 5000×2 is written before the exponent, to get the correct answer, the exponent has to be evaluated **before** the multiplication. In fact, the full order of operations agreement is as follows:

The Full Order of Operations Agreement

- Evaluate any expression inside grouping symbols or **parenthesis** first.
- Perform all **exponents** next.
- Perform all **multiplications** and divisions from left to right next.
- Finally, perform all **additions** and **subtractions** from left to right last.

To help remember the order, someone came up with a little mneumonic, "**P**lease **E**xcuse **M**y **D**ear **Aunt S**ally", or **PEMDAS** for short. The problem with this, is that although it's easy to remember, it doesn't include the "from left to right" part, and many students think that the phrase "My Dear" means that multiplication always comes before division, and that "Aunt Sally" means that addition always comes before subtraction. This is not the case! Multiplication and division are weighted equally, and must be performed from left to right. Similarly, addition and subtraction are weighted equally, and must be performed from left to right.

Exercises: For the following expressions, state the order that the operations must be performed. No need to find the value!

Example: $5 - 3 + 8 \times 2$

Solution: The multiplication comes first, then the subtraction, then the addition.

1. $2 + 3^2$

2. $4 + 0.5 \div 0.25 \times 8$

3. $3(5 - 1)^2$

4. $\frac{1}{2} \times \frac{3}{4} + \left(\frac{3}{2}\right)^3$

5. $24 \div (3 \times \frac{1}{2}) + 4.25 \times 6$

Exercises: For each problem below, write a mathematical expression that answers the question, then use order of operations to find the answer.

1. You went to the Giant's game, and paid $24 for your seat and $20 for parking. Since they were winning, you felt generous and began to buy peanuts for the people around you at $3.25 per bag. Including your ticket and parking, how much had you spent altogether after buying:
   (a) 2 bags of peanuts
   (b) 5 bags of peanuts
   (c) 7 bags of peanuts
   (d) 10 bags of peanuts
2. Your cell phone company charges $35 per month plus overage charges. The deal is, you get 400 anytime minutes covered by your monthly fee, but are charged $0.42 per minute for each minute over 400. What is the monthly charge if your total minutes for the month is:
   (a) 125 minutes
   (b) 475 minutes
   (c) 522 minutes
   (d) 418 minutes

3. As coach of the girl's little league softball, you are responsible for buying the equipment and uniforms for the team. (Don't worry; the parents will pay you back!) Each girl will need a cap that costs $9 and a jearsey that costs $19.45. The team also need to buy a pack of softballs which costs $32.80 and 3 bats at $8.35 each. Excluding tax, how much will you need to spend if the team has:

   (a) 9 girls

   (b) 10 girls

   (c) 11 girls

   (d) 12 girls

4. For the situation in the previous problem, if the girls' families split the total cost evenly (they can't pay the coach, but they don't want them to have to pay!), how much will each family have to pay (to the nearest penny) if the team has:

   (a) 9 girls

   (b) 10 girls

   (c) 11 girls

   (d) 12 girls

Ratio

A ratio is a comparison between two quantities using division. The things being compared may be of the same type. For example, if Sara is playing basketball and makes 12 baskets out of the 19 shots she took, her ratio of baskets made to baskets attempted is $\frac{12}{19}$ which is sometimes written $12 : 19$ (This is read out loud as, "Twelve to nineteen").

Sometimes ratios compare two different types of things. For example, if Jerry buys 8 pounds of apples for $16, then the ratio of dollars to pounds is $\frac{16}{8} = \frac{2}{1}$ or 2. Notice that these ratios give us something more than just a way of writing the numbers in a fraction. In the second case, $\frac{16 \text{ dollars}}{8 \text{ pounds}}$, gives us the rate at which Jerry pays for apples, namely sixteen dollars for every 8 pounds or equivalently, $2 per pound.

Find the following ratios and simplify them whenever possible:

1. If you travel 400 miles in 8 hours, the ratio of miles to hours is: ____________

   What does this ratio tell you?

2. If you travel 352 miles on 16 gallons of gas, the ratio of miles to gallons is: ____________

   What does this ratio tell you?

3. If you work for 28 hours and make $392, the ratio of dollars to hours is: ____________

   What does this ratio tell you?

4. If you make $50,000 in 12 months, what is the ratio of dollars to months? ____________

   What does this ratio tell you?

Equivalent Fractions

We will often find it useful to be able to generate equivalent fractions with a particular denominator. For example, if we want to write $\frac{5}{8}$ as a fraction out of 56 rather than 8, we will need a procedure. Let's see if we can describe how to do this.

Exercises:

1. Find the missing numbers in the fractions below.

(a) $\frac{3}{5} = \frac{\quad}{20}$ (b) $\frac{4}{9} = \frac{\quad}{54}$ (c) $\frac{1}{12} = \frac{\quad}{60}$ (d) $\frac{7}{13} = \frac{\quad}{91}$

2. Describe your procedure for finding the missing numbers in the fractions above.

3. Use your procedure in (2) to help you find the missing number in the fraction: $\frac{4}{5} = \frac{\quad}{42}$

Proportion

When two ratios are equal it is called a proportion.
Example: Suppose that the ratio of boys to girls at a local high school, $\frac{882}{1029}$ is equivalent (after simplifying) to the ratio of boys to girls in an algebra class $\frac{6}{7}$. Then we say that the ratio of boys to girls at the school is in proportion to the number of boys and girls in the class $\longrightarrow \frac{882}{1029} = \frac{6}{7}$. Similarly suppose the ratio of boys to girls in Cleveland is in proportion to the number of boys and girls in that class. If there are 126,000 girls in Cleveland, how many boys are there?

We would write $\dfrac{\text{\# boys in class}}{\text{\# girls in class}} = \dfrac{\text{\# boys in Cleveland}}{\text{\# girls in Cleveland}}$

$$\longrightarrow \frac{6}{7} = \frac{?}{126000}$$

Since $126,000/7 = 18000$, you multiply the numerator by $18,000$:

$$\frac{6 \overset{\times 18000}{\longrightarrow}}{7 \underset{\times 18000}{\longrightarrow}} = \frac{108000}{126000}$$

So there are 108,000 boys in Cleveland.

Exercises:
Use a proportion to solve each problem.

1. A car travels 224 km on 4 gallons of gas. How far can it be expected to travel on a tank of 12 gallons?

2. If you make $600 (after taxes) working for 70 hours, how much will you make working for 500 hours?

3. A 425 pound motorcycle weighs 68 pounds on the moon. How much will a 120 pound woman weigh on the moon?

   How much would you weigh on the moon?

4. 1 inch is equivalent in length to 2.54 centimeters. How tall (in inches) is someone who is 180cm?

5. A recipe for six dozen cookies calls for $2\frac{1}{2}$ cups of flour. How many cups of flour are needed for 10 dozen cookies?

6. The floor plan of a house is drawn to the scale of $\frac{1}{4}'' = 1'$. The master bedroom measures $3\frac{1}{2}''$ by $5''$ on the blueprints. What is the actual size of the room?

7. On average, an adult flea is 3mm long but it can long jump 330mm (about 13 inches) and high jump 204mm (about 8 inches). If you could jump in proportion with a flea, how far could you long jump? High jump?

8. If a company's stock falls by 3% or $1.20, how much was the stock worth originally?

9. a) If one gallon of paint covers 400 square feet, how many gallons of paint do you need to cover 5000 square feet?

b) If one gallon of paint covers 400 square feet, how many square feet will 27 gallons cover?

10. If your car gets 25 miles to the gallon and gas costs $3.40 per gallon, how much will a 420 mile trip cost?