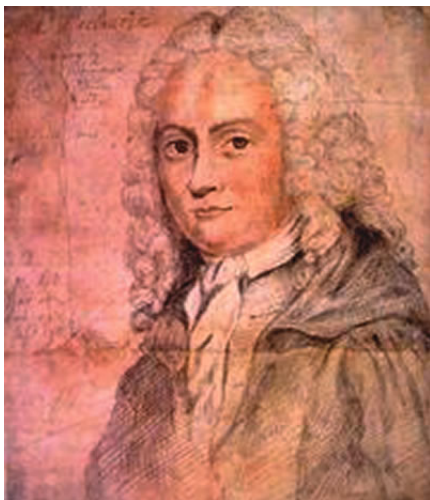## Determinants

Leibniz seems to have introduced the idea of determinants (though not in name) before the end of the 17th century.

He used them much as we do in algebra to solve systems of equations.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Determinants

Mclaurin and Cramer made more thorough work of determinants - all long before matrices were first defined.

The larger the determinant, however, the more intimidating the task of evaluating it . . .

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

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$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

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$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Equivalently:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{matrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{matrix}$$

$$- \quad - \quad - \qquad + \quad + \quad +$$

Example:

$$\begin{vmatrix} 5 & -4 & 3 \\ 2 & 1 & 6 \\ 8 & 3 & 9 \end{vmatrix} = \begin{matrix} 5 & -4 & 3 & 5 & -4 \\ 2 & 1 & 6 & 2 & 1 \\ 8 & 3 & 9 & 8 & 3 \end{matrix}$$

$$\text{-24} \quad \text{-90} \quad \text{+72} \quad \text{+45} \quad \text{-192} \quad \text{+18}$$

$$= \text{-171}$$

A Little History . . .

William Rowan Hamilton
(1805 – 1865)
Dublin

Complex Numbers $(a + bi), i = \sqrt{-1}$

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$$(a + bi)(c + di)$$

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$$(a + bi)(c + di)$$
$$= (ac - bd) + (ad + bc)i$$

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Hamilton: $\mathbb{C} = \mathbb{R}^2$

$$\text{so } a + bi \Rightarrow (a, b)$$
$$\text{and } (a + bi)(c + di) \Rightarrow (a, b)(c, d) = (ac - bd, ad + bc)$$

What about multiplication in $\mathbb{R}^3$?

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Good question.

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What are properties we've come to expect of multiplication?

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- and . . .

Unfortunately . . .

Creating a rule for multiplication in $\mathbb{R}^3$ that retained the properties and consequences of multiplication found in $\mathbb{R}$ and $\mathbb{R}^2$ proved elusive.

Eventually (16 years after he began his pursuit), Hamilton had an epiphany . . .

Quaternions

If $\mathbb{R}^3$ won't comply, why not consider $\mathbb{R}^4$?

Hamilton described numbers of the form $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ where a was called the real or scalar part and $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ the vector or imaginary part.

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His epiphany?

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

. . . of course.

Quaternions

Hamilton's rules for quaternion multiplication are involved:

- $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$

- $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$

- $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$

- $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$

. . and while not commutative, *skew symmetry* was apparently close enough.

The set $\mathbb{R}^4$ with Hamilton's quaternion multiplication is usually denoted $\mathbb{H}$.

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Consider the length of time Hamilton spent puzzling about this question and ask yourself how long you'd be willing to pursue the answer . . .

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge
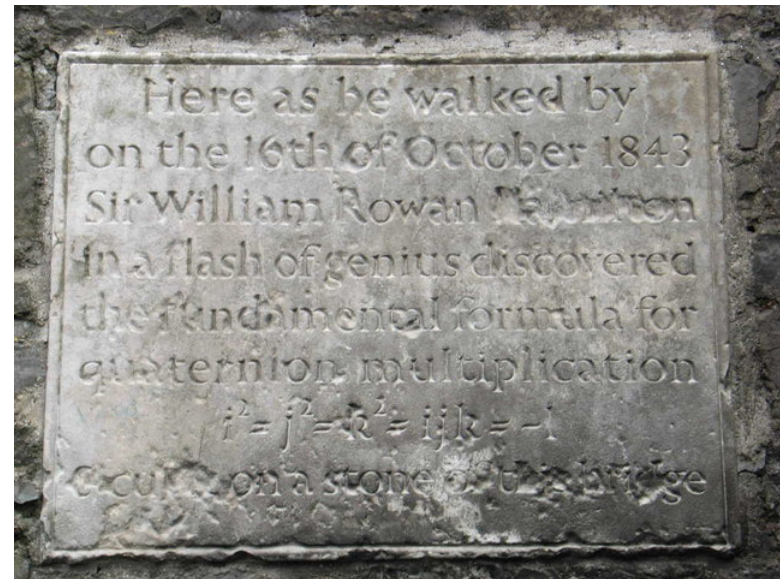
As luck would have it quaternions proved too cumbersome for most
(which may explain why Gauss, who had discovered many of the same results in 1819, never published
his observations) and it wasn't until one of Hamilton's students, Peter Tait, found himself playing with
the numbers that vector multiplication found its way into the math books.
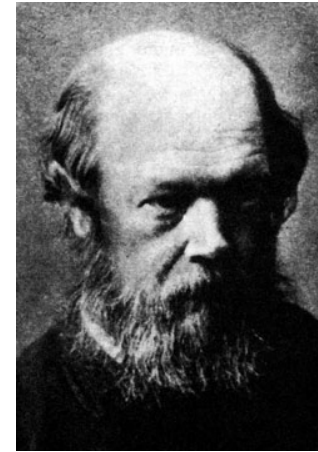
Tait considered the product of $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\vec{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$
(note the absence of the fourth dimension).
His results were

$$(\vec{v})(\vec{w}) = -(v_1w_1 + v_2w_2 + v_3w_3) + (v_2w_3 - v_3w_2)\mathbf{i} + (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}$$

. . . and following Hamilton's lead, Tait proposed that vector multiplication in $\mathbb{R}^3$ could be separated into two components . . .

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The scalar product: $\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3$

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The scalar product: $\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3$

and the vector product: $\vec{v} \times \vec{w} = (v_2w_3 - v_3w_2)\mathbf{i} + (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}$

Dot Product

For $\vec{v} = (v_1, v_2, ..., v_n)$, $\vec{w} = (w_1, w_2, ..., w_n)$, $\vec{v} \cdot \vec{w} = \sum_{k=1}^{n} v_k w_k$

And equivalently $\vec{v} \cdot \vec{w} = ||\vec{v}|| ||\vec{w}|| \cos\theta$

The geometric definition is often used to determine the angle between two vectors:

$$\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| ||\vec{w}||}\right)$$

. . . and to determine the work done in applying a force in the direction an object moves.
$W = \vec{F} \cdot d$
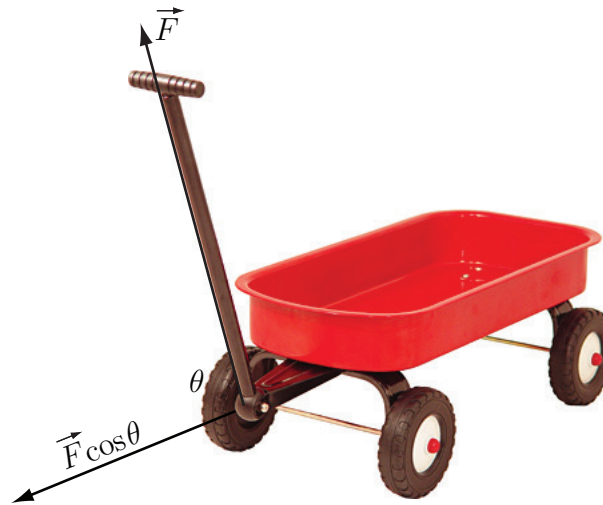
Cross Product

For $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\vec{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

And equivalently $\vec{v} \times \vec{w} = (\|\vec{v}\|\|\vec{w}\| \sin \phi)\mathbf{n}$, where $\mathbf{n}$ is the unit vector normal to $\vec{v}$ and $\vec{w}$ and obeying the "right hand" rule.

$(\|\vec{v}\|\|\vec{w}\|\sin(\phi))\vec{\mathbf{n}}$
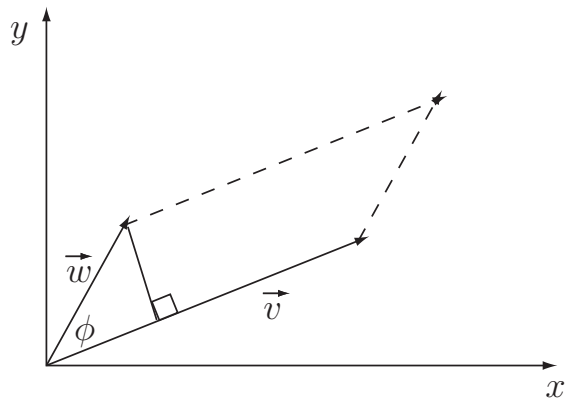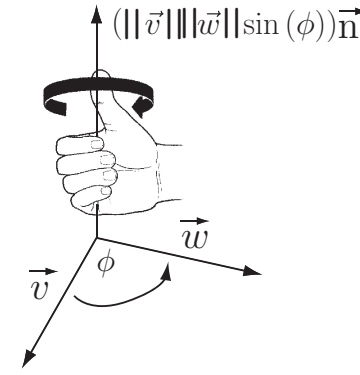
The magnitude in the geometric definition is the area of a parallelogram with sides $\vec{v}$ and $\vec{w}$.

$A = bh$
$\quad = \|\vec{v}\|\|\vec{w}\|\sin(\phi)$

Cross Product

A few remarks about the relationship between the analytical definition and the geometric.

Consider $\vec{v} \times \vec{w}$ in $\mathbb{R}^2$ (since $\vec{v}$ and $\vec{w}$ occur in a common plane this is a reasonable simplification).

We want to show that for $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$

$$||\vec{v} \times \vec{w}|| = \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = ||\vec{v}||\,||\vec{w}|| \sin \phi$$

We already see that the area of the parallelogram with sides $\vec{v}$ and $\vec{w}$ is given by $||\vec{v}||\,||\vec{w}|| \sin \phi$.

$$A = bh$$
$$= ||\vec{v}||\,||\vec{w}|| \sin (\phi)$$
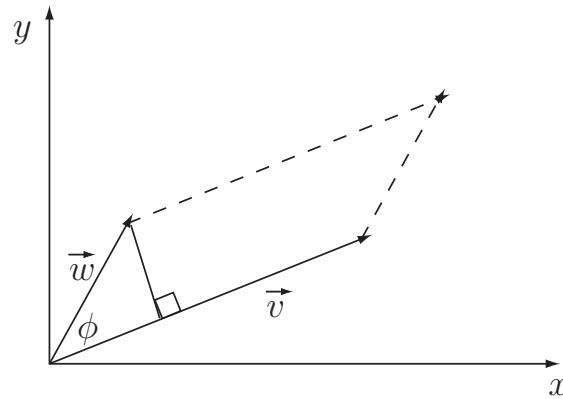
Cross Product

Now consider the coordinate form of the vectors.
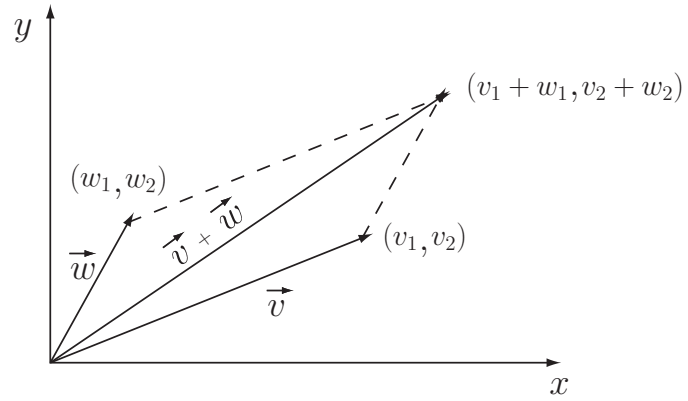
Cross Product

The area of the enclosing rectangle is given by $(v_1 + w_1)(v_2 + w_2) = v_1v_2 + v_1w_2 + v_2w_1 + w_1w_2$.
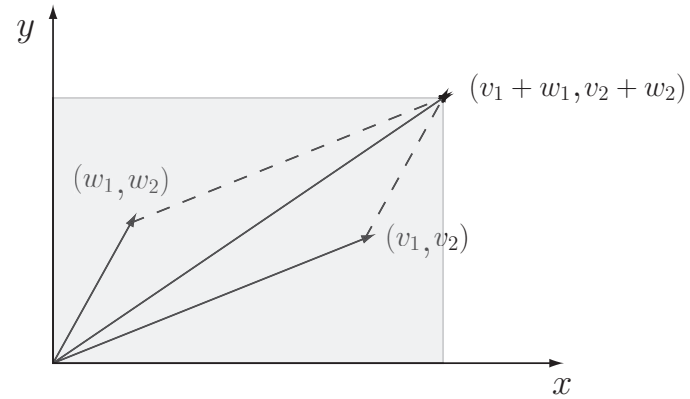
Area: $(v_1 + w_1)(v_2 + w_2) = v_1v_2 + v_1w_2 + v_2w_1 + w_1w_2$

Cross Product

The areas of the surrounding triangles:
$\frac{1}{2}w_1w_2$ and $\frac{1}{2}v_1v_2$.
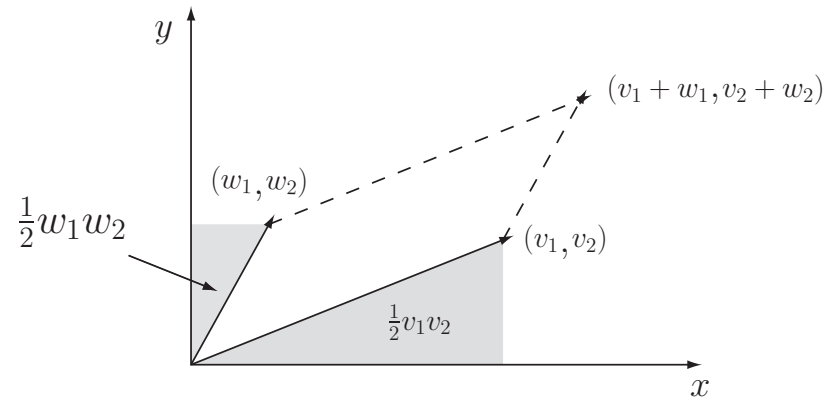
Cross Product

The areas of the surrounding triangles:
$\frac{1}{2}w_1w_2$ and $\frac{1}{2}v_1v_2$.

And the surrounding trapezoids:
$v_2 \cdot \frac{1}{2}(w_1 + w_1 + v_1)$ and $w_1 \cdot \frac{1}{2}(v_2 + v_2 + w_2)$
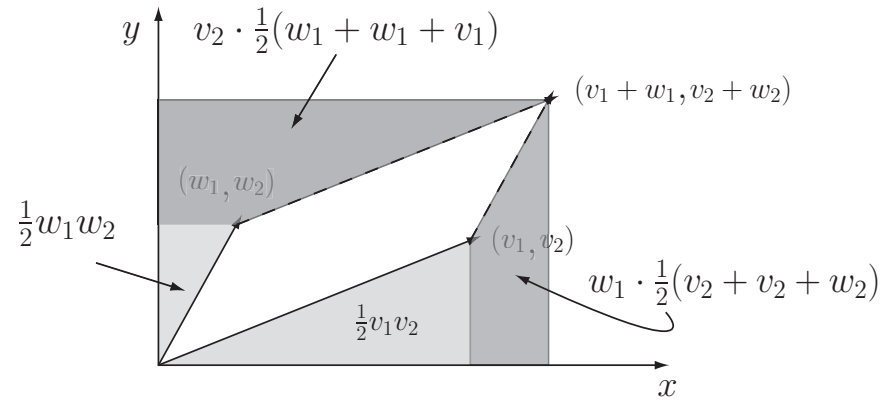
Cross Product

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And the surrounding trapezoids:
$v_2 \cdot \frac{1}{2}(w_1+w_1+v_1)$ and $w_1 \cdot \frac{1}{2}(v_2+v_2+w_2)$

Which totals to $w_1w_2 + v_1v_2 + 2w_1v_2$.
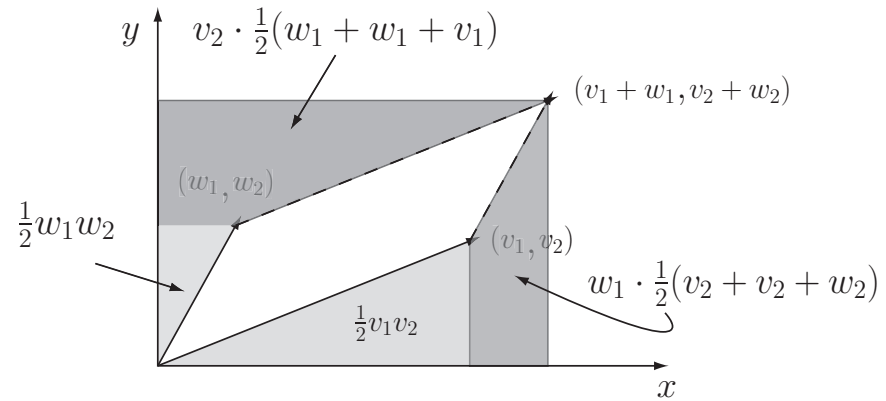
Total: $\frac{1}{2}w_1w_2 + \frac{1}{2}v_1v_2 + v_2 \cdot \frac{1}{2}(w_1+w_1+v_1) + w_1 \cdot \frac{1}{2}(v_2+v_2+w_2)$

$= \frac{1}{2}w_1w_2 + \frac{1}{2}v_1v_2 + w_1v_2 + w_1v_2 + \frac{1}{2}v_1v_2 + \frac{1}{2}w_1w_2$

$= w_1w_2 + v_1v_2 + 2w_1v_2$

Cross Product

It follows that the area of the parallelogram is given by

$$A = v_1 v_2 + v_1 w_2 + v_2 w_1 + w_1 w_2 - (w_1 w_2 + v_1 v_2 + 2 w_1 v_2)$$
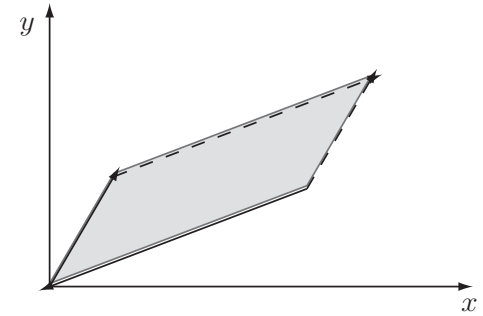
$$= v_1 w_2 - v_2 w_1$$

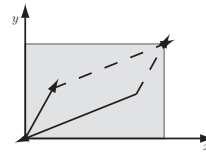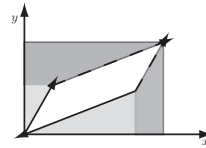$$v_1 v_2 + v_1 w_2 + v_2 w_1 + w_1 w_2$$

$$w_1 w_2 + v_1 v_2 + 2 w_1 v_2$$

Parallelogram Area: $v_1 v_2 + v_1 w_2 + v_2 w_1 + w_1 w_2 - (w_1 w_2 + v_1 v_2 + 2 w_1 v_2)$

$$= v_1 w_2 - v_2 w_1$$

Cross Product

It follows that the area of the parallelogram is given by

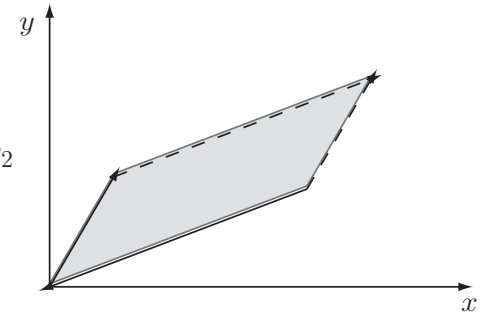$$A = v_1v_2 + v_1w_2 + v_2w_1 + w_1w_2 - (w_1w_2 + v_1v_2 + 2w_1v_2)$$

$$= v_1w_2 - v_2w_1$$

$$= \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$v_1v_2 + v_1w_2 + v_2w_1 + w_1w_2$$

$$w_1w_2 + v_1v_2 + 2w_1v_2$$

Parallelogram Area: $v_1v_2 + v_1w_2 + v_2w_1 + w_1w_2 - (w_1w_2 + v_1v_2 + 2w_1v_2)$
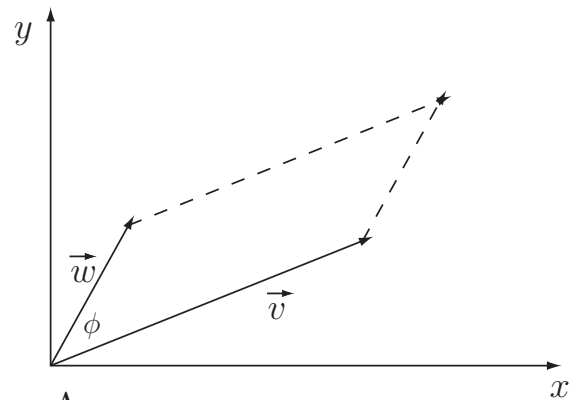
$$= v_1w_2 - v_2w_1$$

$$= \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

Cross Product

We conclude that

$$||\vec{v} \times \vec{w}|| = \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = ||\vec{v}|| \, ||\vec{w}|| \sin \phi$$
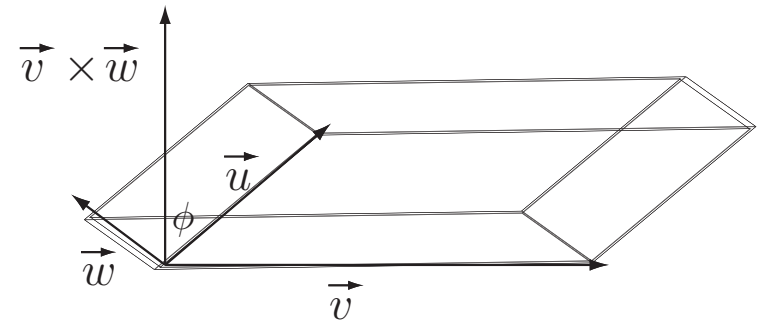

Area
$$= ||\vec{v}|| \, ||\vec{w}|| \sin(\phi)$$
$$= \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

The Scalar Triple Product

The volume of a parallelepiped with edge vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is given by the Scalar Triple Product where the angle ϕ is less than π (so $\vec{u}$ lies in the same vertical direction as $\vec{v} \times \vec{w}$.)

$$V = |(\vec{v} \times \vec{w}) \cdot \vec{u}|$$

Cross Product

$$\vec{v} = \vec{r} \times \vec{s}$$

Consider a rigid spinning body with its center of mass at the origin. The vector $\vec{r}$ represents its spin which lies on the axis of rotation and emanates from the origin. We assume the spin is counterclockwise (as seen from $\vec{r}$ looking toward the origin. The magnitude $||\vec{r}||$ represents the angular velocity (in radians/sec e.g.) – the faster the spin the longer the vector.
Let $\vec{s}$ be the displacement vector from the origin to any point on the body and let θ be the angle between $\vec{r}$ and $\vec{s}$. As point s moves around the axis it has (linear) velocity $\vec{v}$. The magnitude $||\vec{v}||$ is $||\vec{r}||$ times the distance s is from the axis. In other words $||\vec{v}|| = ||\vec{r}||||\vec{s}|| \sin \theta$. Since $\vec{v}$ is tangent to the surface at s it is perpendicular to the plane containing $\vec{r}$ and $\vec{s}$ so it meets the definition of $\vec{v} = \vec{r} \times \vec{s}$.
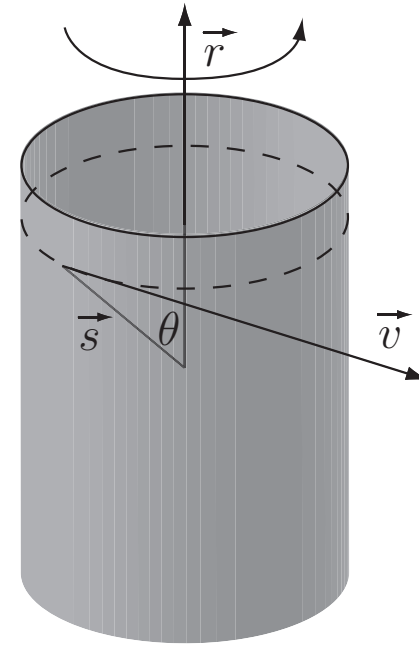
Example

Find the vector parallel to the line of intersection of the planes $2x + 3y + z = 4$ and $3x - 4y - 3z = 5$.

$2x + 3y + z = 4$
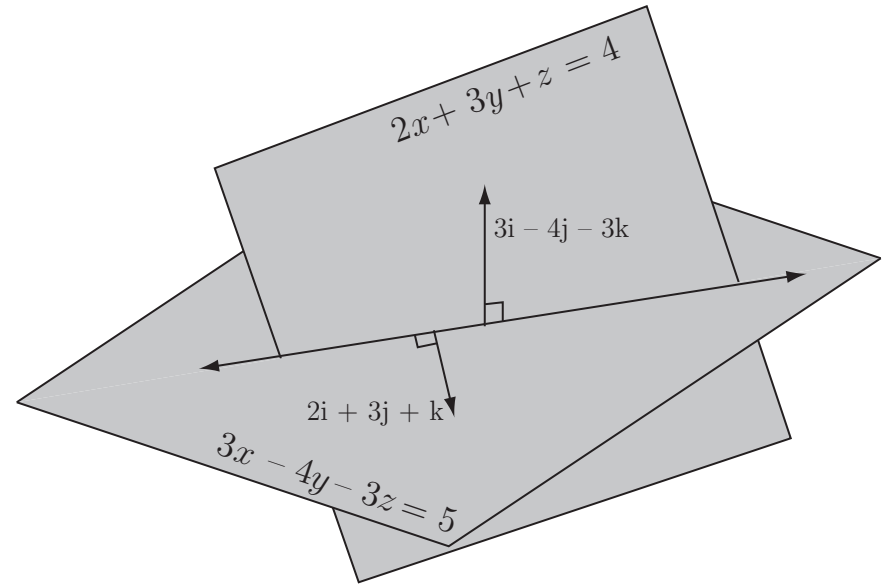
$3i - 4j - 3k$

$2i + 3j + k$
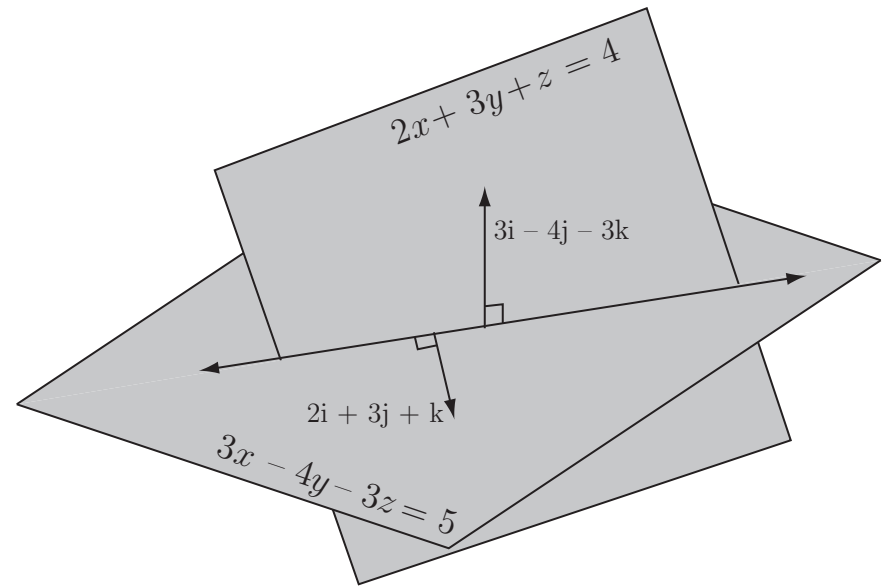
$3x - 4y - 3z = 5$

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Find the vector parallel to the line of intersection of the planes $2x + 3y + z = 4$ and $3x - 4y - 3z = 5$.

Solution: Since the line of intersection lies in both planes it is perpendicular to both the defining normal vectors. That is, it lies in the direction of the vector given by $(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \times (3\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 3 & -4 & -3 \end{vmatrix} = -5\mathbf{i} + 9\mathbf{j} - 17\mathbf{k}.$$

Since the line is also parallel to $(3\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}) \times (2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$ (vector pointed in the opposite direction) we can also have $5\mathbf{i} - 9\mathbf{j} + 17\mathbf{k}$ as a solution.

Distance from a point to a plane

Consider the plane $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$
(equivalently $Ax + By + Cz - D = 0$). The unit
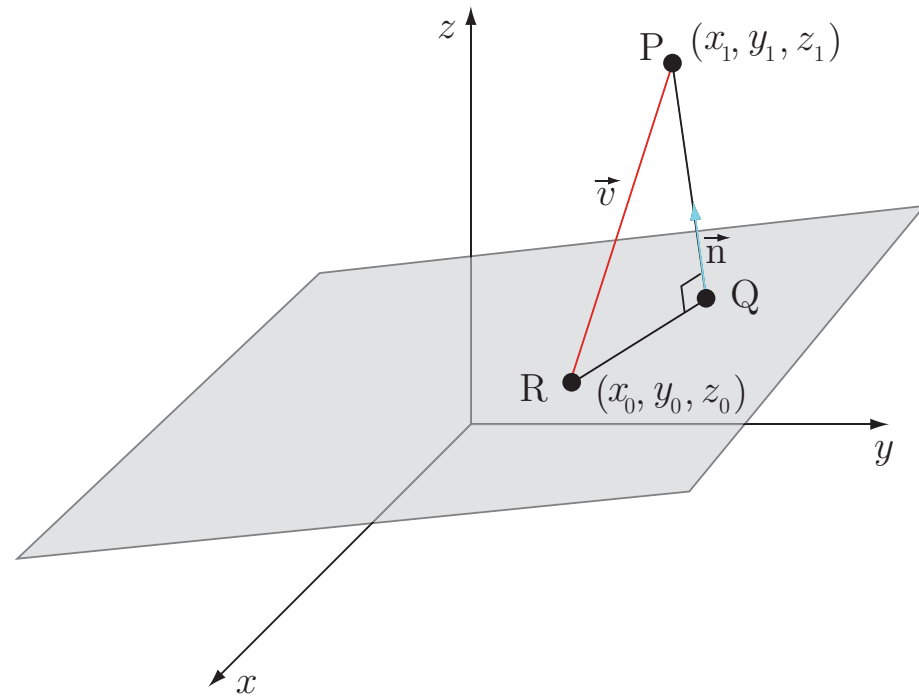normal vector to this plane is given by

$$\mathbf{n} = \frac{A\mathbf{i}+B\mathbf{j}+C\mathbf{k}}{\sqrt{A^2+B^2+C^2}}$$

The distance we want is $||\overrightarrow{PQ}||$ which is the magnitude
of the projection of $\vec{v}=\overrightarrow{RP}$ on $\mathbf{n}$.

It follows that

$$|\vec{v}\cdot\mathbf{n}| = |[(x_1-x_0)\mathbf{i}+(y_1-y_0)\mathbf{j}+(z_1-z_0)\mathbf{k}]\cdot\mathbf{n}|$$

$$= \frac{|A(x_1-x_0)+B(y_1-y_0)+C(z_1-z_0)|}{\sqrt{A^2+B^2+C^2}}$$

Or equivalently $d = \dfrac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$

Distance from a point to a plane

Consider the plane $A(x-x_0)+B(y-y_0)+C(z-z_0) = 0$ (equivalently $Ax + By + Cz - D = 0$). The unit normal vector to this plane is given by

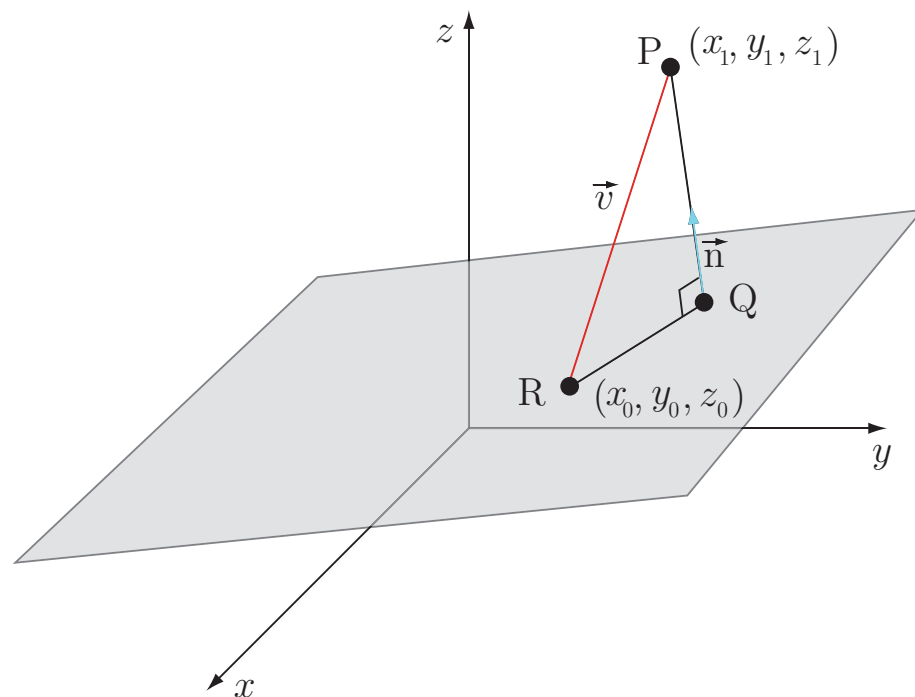$$\mathbf{n} = \frac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}}$$

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$$|\vec{v} \cdot \mathbf{n}| = |[(x_1 - x_0)\mathbf{i} + (y_1 - y_0)\mathbf{j} + (z_1 - z_0)\mathbf{k}] \cdot \mathbf{n}|$$

$$= \frac{|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|}{\sqrt{A^2 + B^2 + C^2}}$$

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Example Find the distance from $P = (2, 0, -1)$ to the plane $3x - 2y + 8z + 1 = 0$.

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Solution: $d = \dfrac{|3(2) - 2(0) + 8(-1) + 1|}{\sqrt{3^2 + (-2)^2 + 8^2}} = \dfrac{|-1|}{\sqrt{77}} = \dfrac{1}{\sqrt{77}}$