Relative Extrema – The Second Derivative Test

$$f(x, y) = ax^2 + bxy + cy^2$$

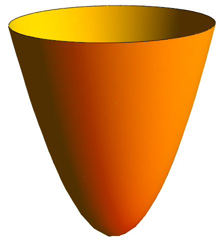


Figure 1: $z = x^2 + y^2$

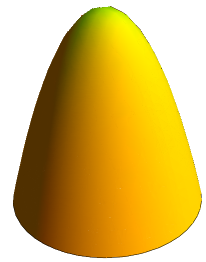


Figure 2: $z = -(x^2 + y^2)$

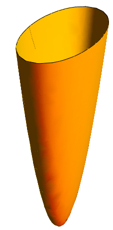


Figure 3: $z = 5x^2 + 2xy + y^2$

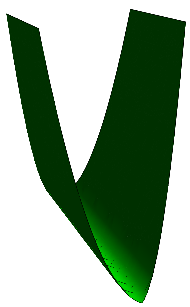


Figure 4: $z = x^2 + 2xy + y^2$

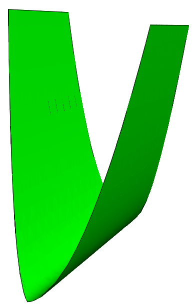


Figure 5: $z = x^2 - 2xy + y^2$

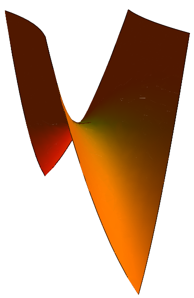


Figure 6: $z = x^2 + 3xy + y^2$

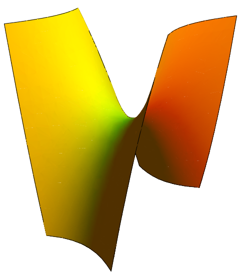


Figure 7: $z = x^2 - y^2$

Relative Extrema

$$\begin{aligned}
f(x,y) &= ax^2 + bxy + cy^2 \\
&= a\left(x^2 + \frac{b}{a}xy + \frac{c}{a}y^2\right) \\
&= a\left(\left(x + \frac{b}{2a}y\right)^2 - \frac{b^2}{4a^2}y^2 + \frac{c}{a}y^2\right) \\
&= a\left(\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)y^2\right) \\
&= a\left(\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)y^2\right)
\end{aligned}$$

Relative Extrema

$$f(x,y) = a\left(\left(x+\frac{b}{2a}y\right)^2+\left(\frac{4ac-b^2}{4a^2}\right)y^2\right)$$

$a > 0$:

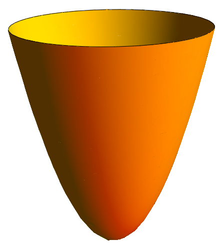


Figure 8: $4ac - b^2 > 0$

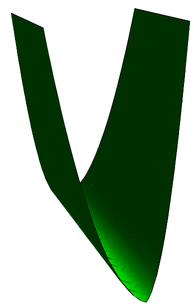


Figure 9: $4ac - b^2 = 0$

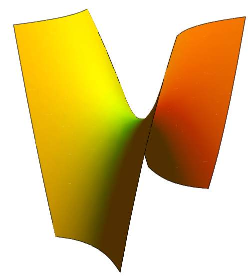


Figure 10: $4ac - b^2 < 0$

$a < 0$:

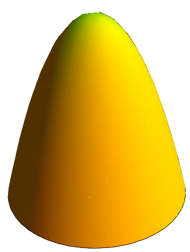


Figure 11: $4ac - b^2 > 0$

Recall the Taylor Polynomial of degree 2 centered at (a, b) is given by:

$$Q(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2$$

Now suppose (x_0, y_0) is a critical point of $f(x, y)$ – so $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

Then the Taylor Polynomial will have the form:

$$Q(x,y) = f(x_0,y_0) + \frac{f_{xx}(x_0,y_0)}{2}(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{f_{yy}(x_0,y_0)}{2}(y-y_0)^2$$

. . . Which is quadratic of the form $f(x, y) = ax^2 + bxy + cy^2$ (with a vertical shift).

It follows that near (x_0, y_0) we can write

$$f(x,y) \approx f(x_0,y_0) + \frac{f_{xx}(x_0,y_0)}{2}(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{f_{yy}(x_0,y_0)}{2}(y-y_0)^2$$

Since $f(x,y)$ is approximated near (x_0, y_0) by

$$f(x,y) \approx f(x_0,y_0) + \frac{f_{xx}(x_0,y_0)}{2}(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{f_{yy}(x_0,y_0)}{2}(y-y_0)^2$$

We conclude that in the vicinity of (x_0, y_0), $f(x,y)$ will look like one of the quadric surfaces:

$a > 0:$

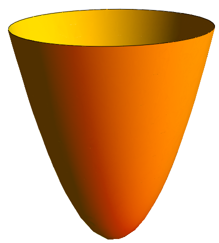


Figure 12: $4ac - b^2 > 0$

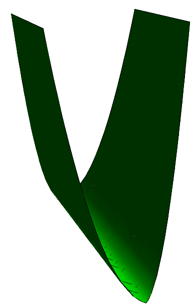


Figure 13: $4ac - b^2 = 0$

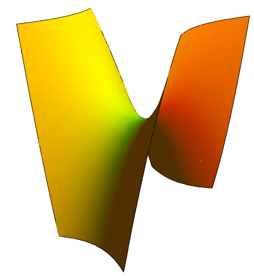


Figure 14: $4ac - b^2 < 0$

$a < 0:$

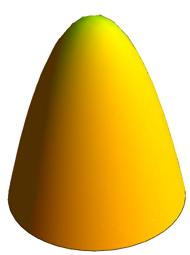


Figure 15: $4ac - b^2 > 0$

Recall the quadric form: $f(x,y) = a\left(\left(x+\frac{b}{2a}y\right)^2 + \left(\frac{4ac-b^2}{4a^2}\right)y^2\right)$

$$f(x,y) \approx f(x_0,y_0) + \frac{f_{xx}(x_0,y_0)}{2}(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{f_{yy}(x_0,y_0)}{2}(y-y_0)^2$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$a(x-x_0)^2 + \qquad b(x_0,y_0)(x-x_0)(y-y_0) + \qquad c(y-y_0)^2$$

Then from $a = \frac{f_{xx}(x_0,y_0)}{2}$, $b = \frac{f_{xy}(x_0,y_0)}{2}$ and $c = \frac{f_{yy}(x_0,y_0)}{2}$ we have

$4ac - b^2 = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - \left(f_{xy}(x_0,y_0)\right)^2$

$4ac - b^2 = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$

So if $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$ then . . .

$f_{xx}(x_0, y_0) > 0$:

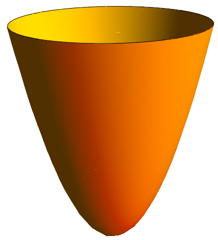


Figure 16: $D > 0$

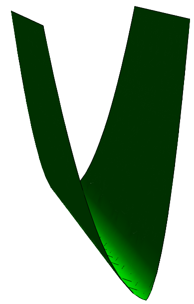


Figure 17: $D = 0$

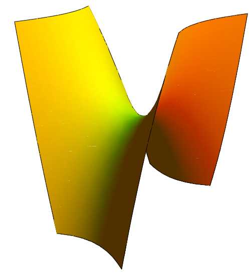


Figure 18: $D < 0$

$f_{xx}(x_0, y_0) < 0$:

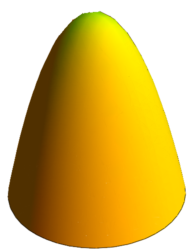
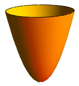


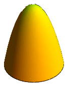
Figure 19: $D > 0$

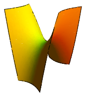
For a critical point of $f(x, y)$ at (x_0, y_0),

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

$$= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

If $D > 0$ and $f_{xx}(x_0, y_0) > 0$ then $f(x_0, y_0)$ is a local minimum. 

If $D > 0$ and $f_{xx}(x_0, y_0) < 0$ then $f(x_0, y_0)$ is a local maximum. 

If $D < 0$ then there is a saddle at (x_0, y_0) os there is neither min nor max. 

If $D = 0$ the test is inconclusive.