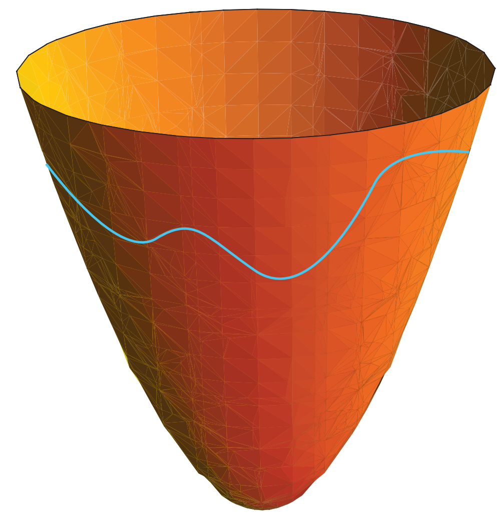
Notes

Lagrange Multipliers: Finding the high and low points of path on a surface.



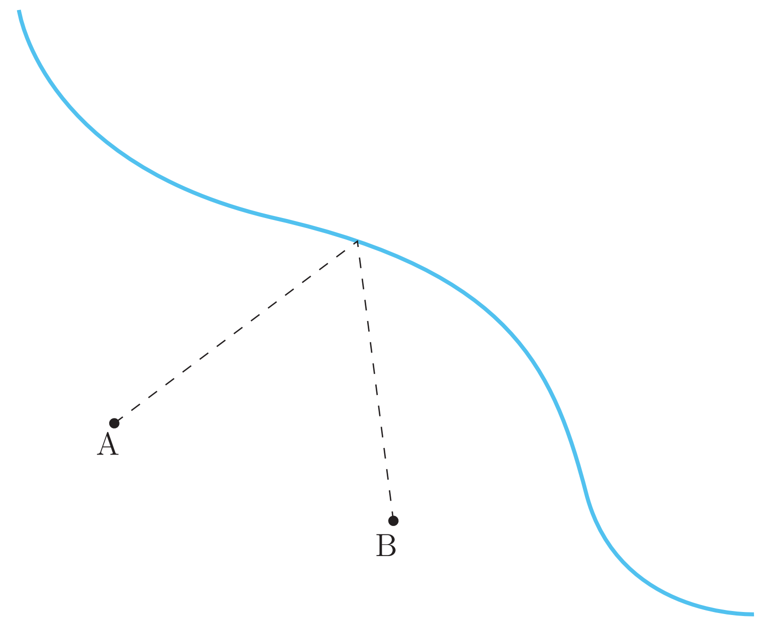
Finding the maximum (or minimum) value of an unrestricted function is often a study in the infinte. The extreme values of that function restricted to a curve, however, will produce more finite if not more interesting results.

While we can *see* the answer, actually finding it requires some insight.

Lagrange Multipliers: Another form of constraint.

We begin with a related if well travelled problem.

A rider leaves point A on his beast of burden and heads for point B. However, he needs to get water for the animal so he detours to the nearby river on his way to B. Where along the river should the rider stop to get water in order to minimize the total distance of the trip?



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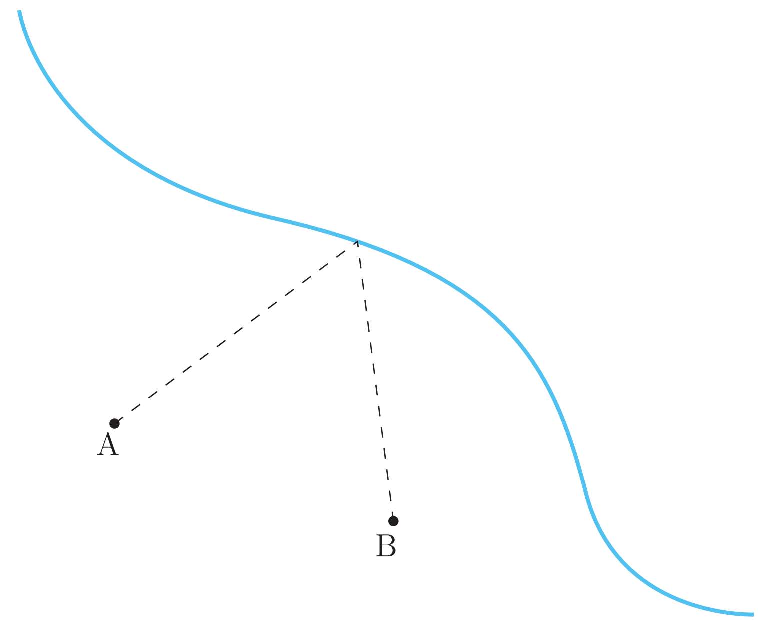
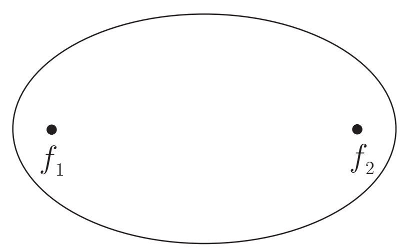


The solution offered by Archimedes, among others, was based on the idea of an ellipse.
An ellipse is defined by two points, called foci.

Ellipse



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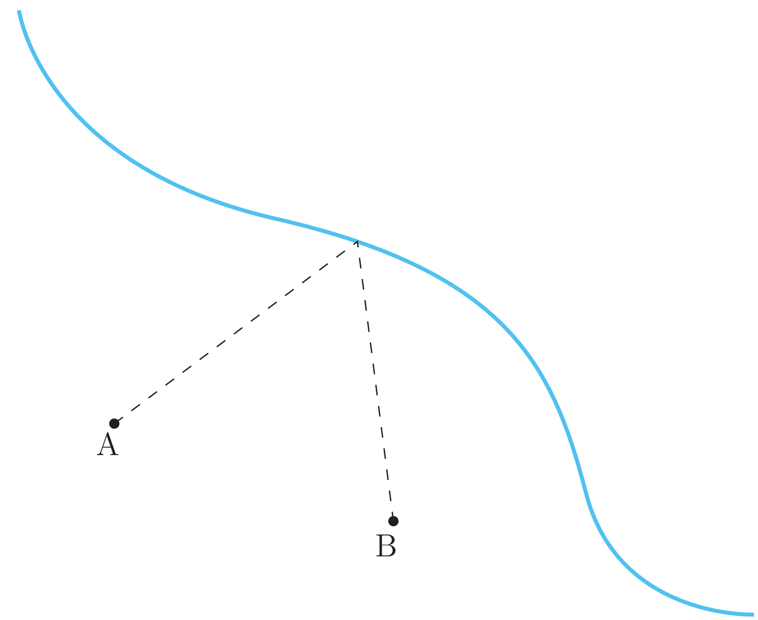

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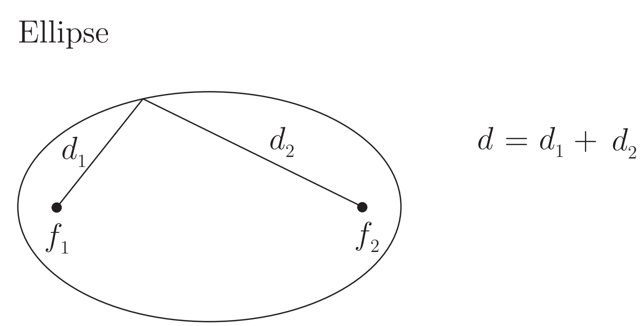

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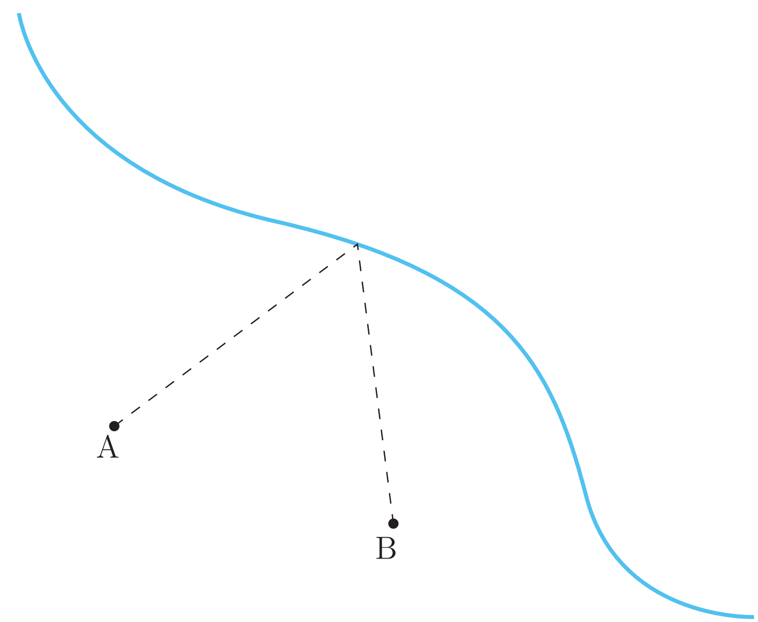

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Each point on the ellipse lies a combined distance $d = d_1 + d_2$ from the foci. And this distance is constant for all points on the ellipse.

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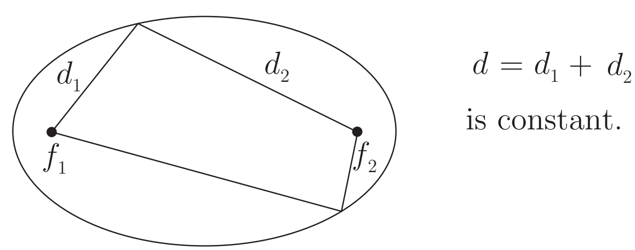
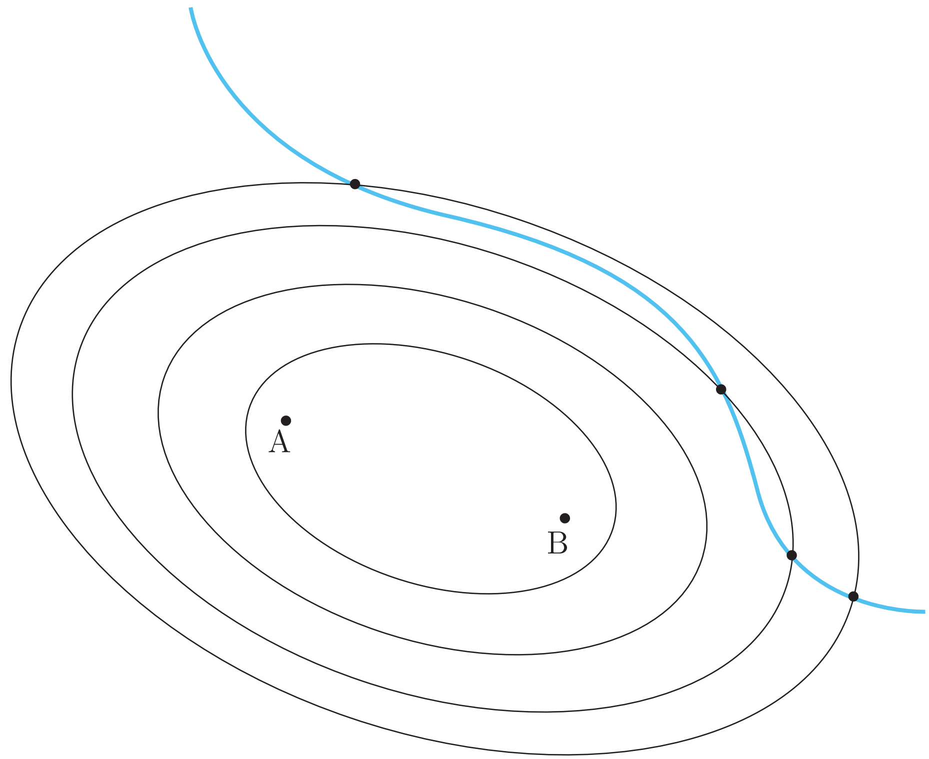


Lagrange Multipliers: Another form of constraint.

Then every point on the river bank lies on some ellipse centered on points A and B. If we can find the ellipse that intersects the river with the smallest value, $d = d_1 + d_2$ then the point (or points) where the ellipse intersects the river is the point where the rider should stop.



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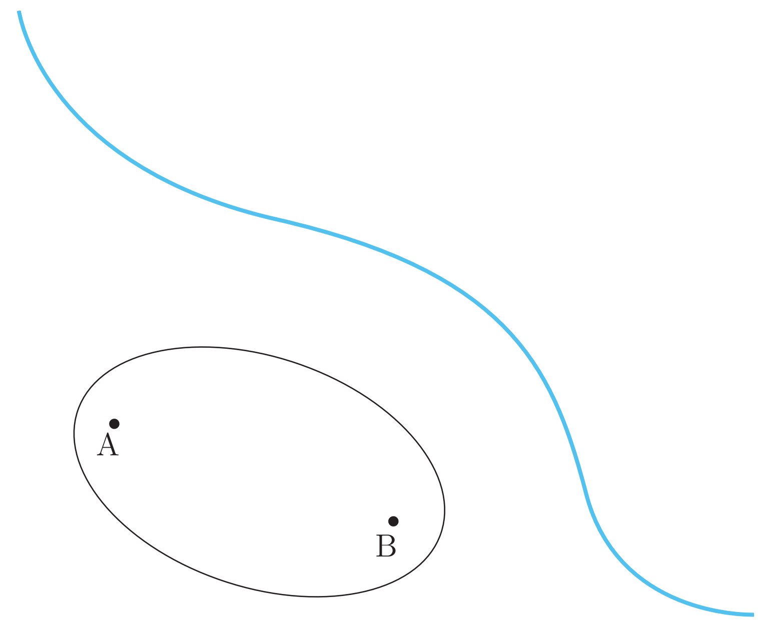
If we begin with one ellipse and expand it . . .



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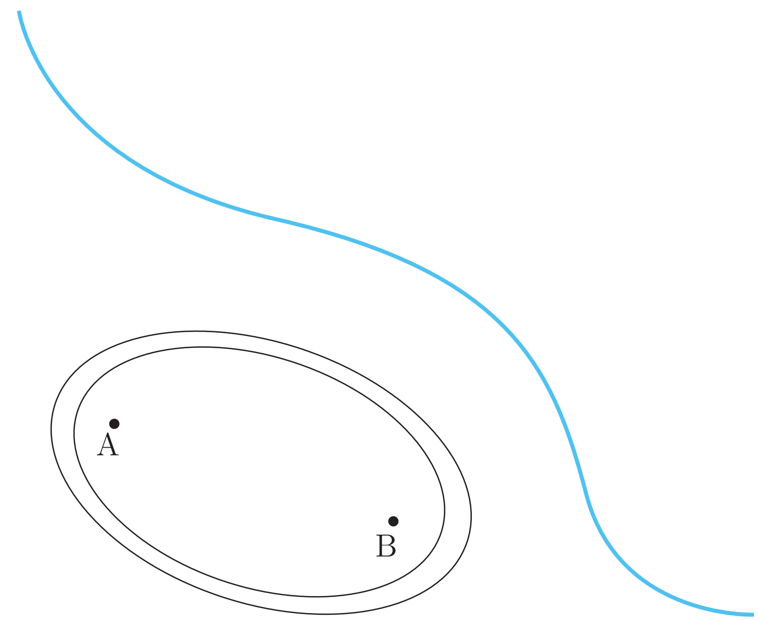
Eventually we get to the river.



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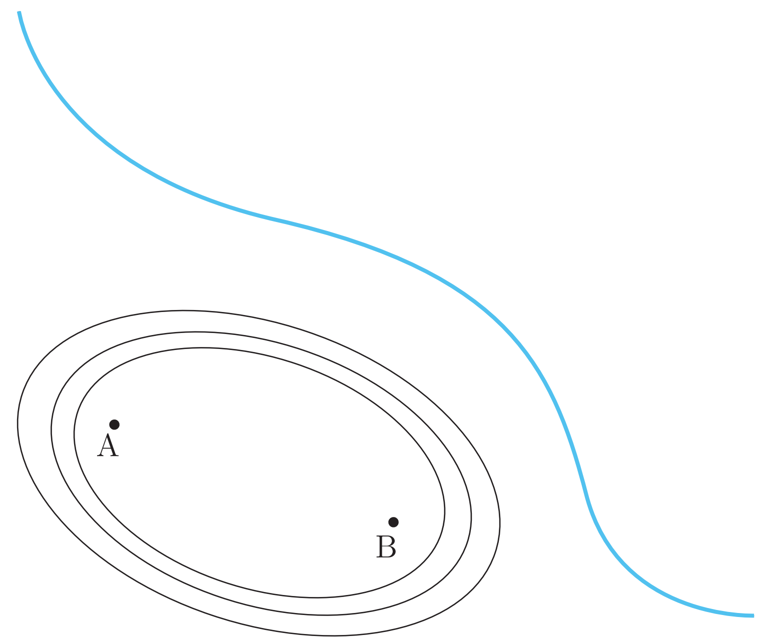

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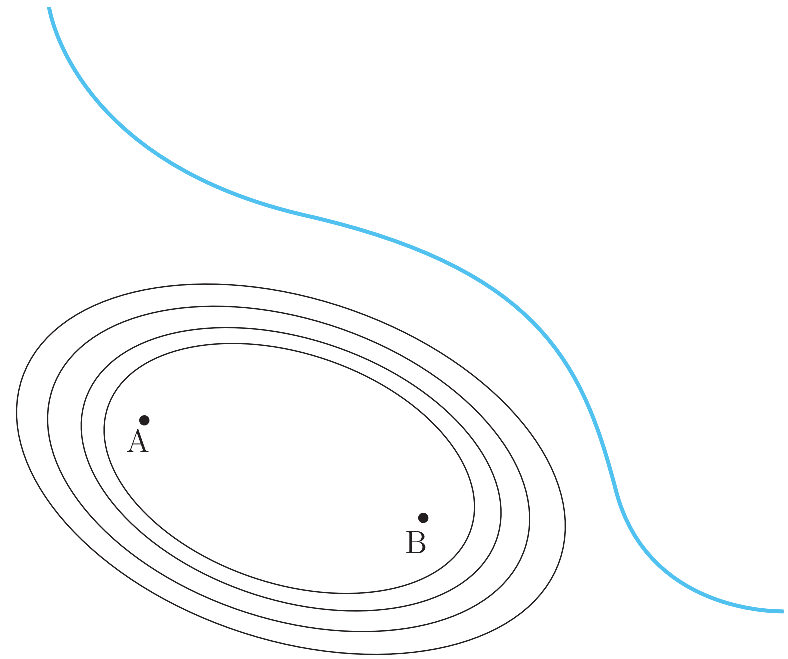

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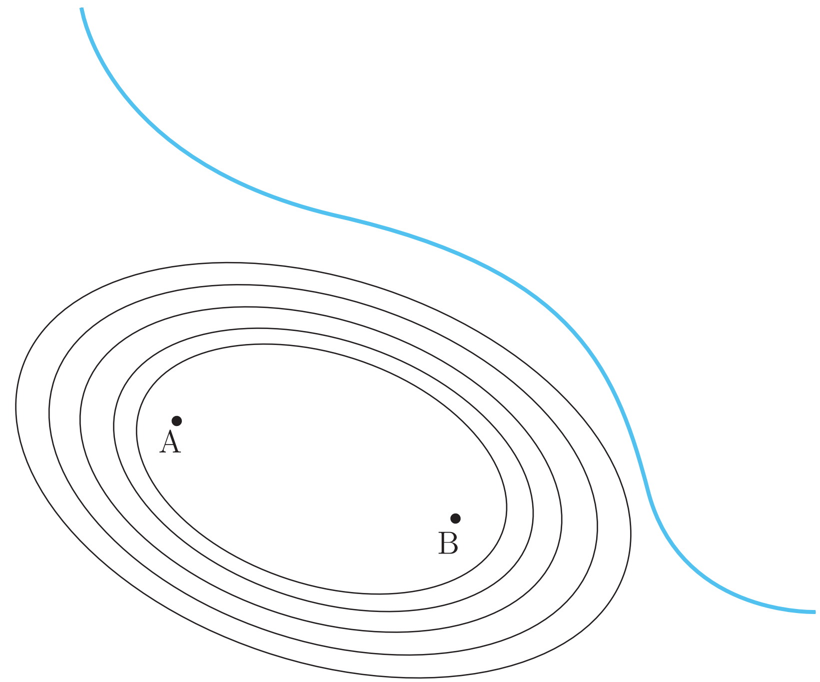

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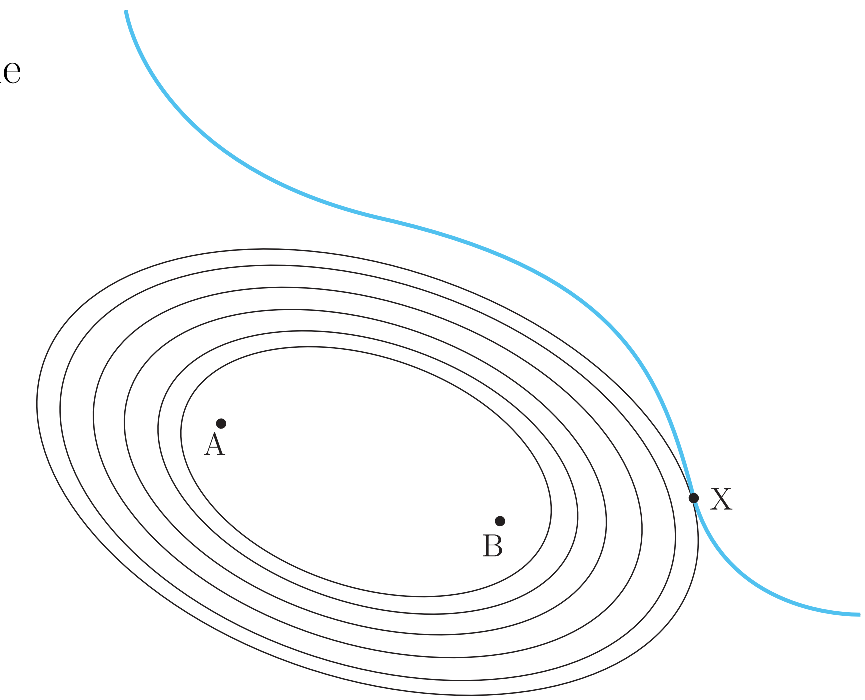
Note that this point occurs where an ellipse is tangent to the river.



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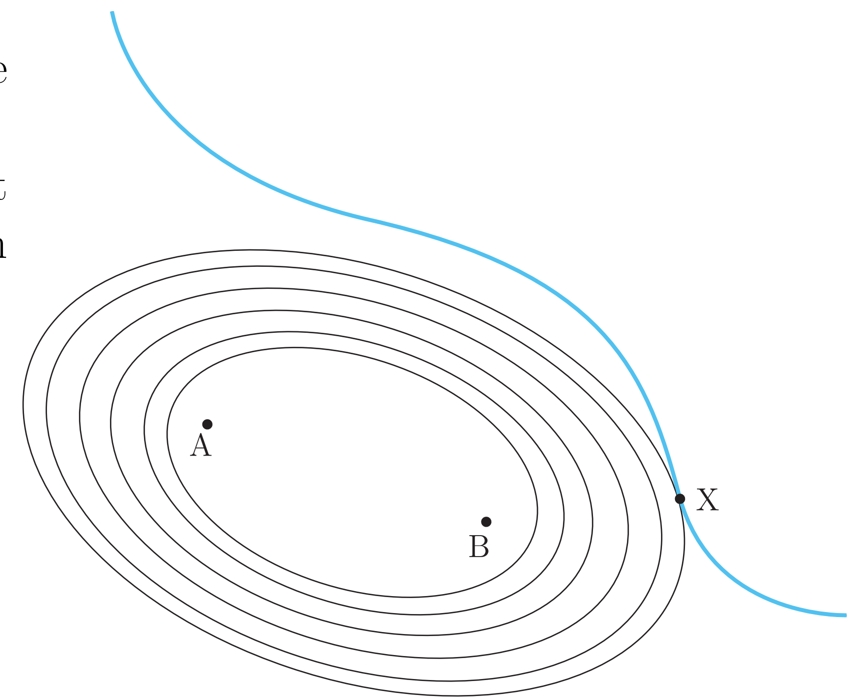
If we begin with one ellipse and expand it . . .
Eventually we get to the river.
Note that this point occurs where an ellipse is tangent to the river.
Also note that since the two curves share a common tangent at this point, the gradient vectors for the functions to which these are level curves are parallel.



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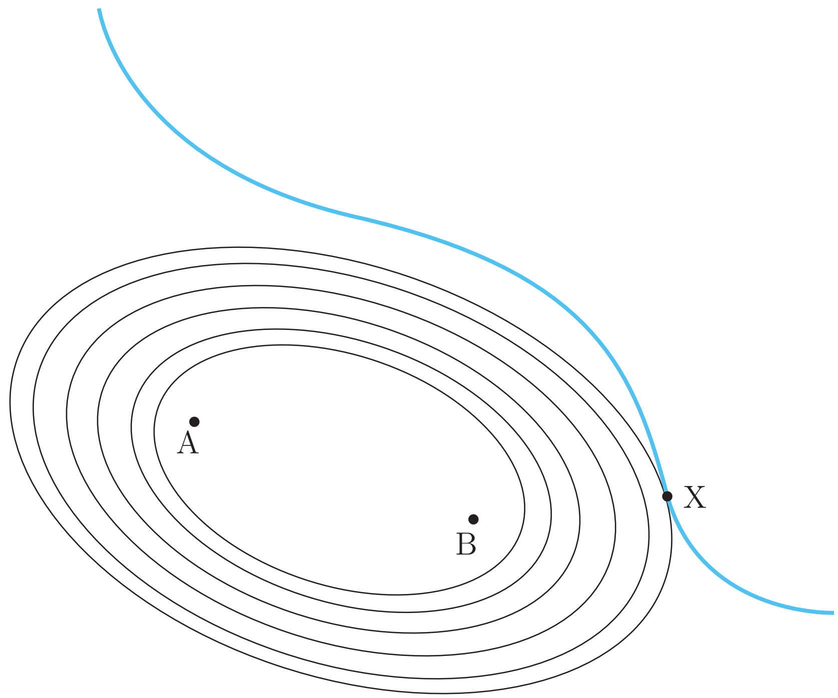
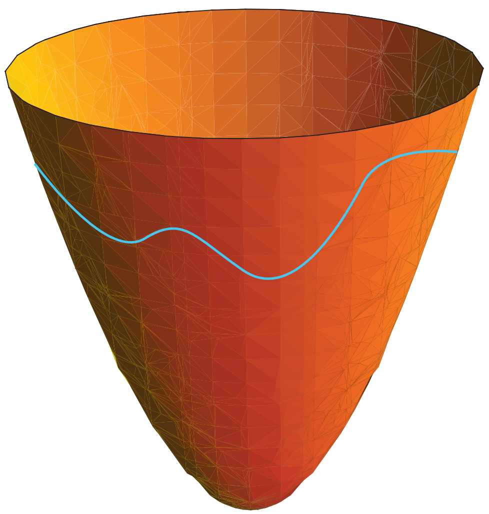
So what does the river problem have to do with the path on the surface?





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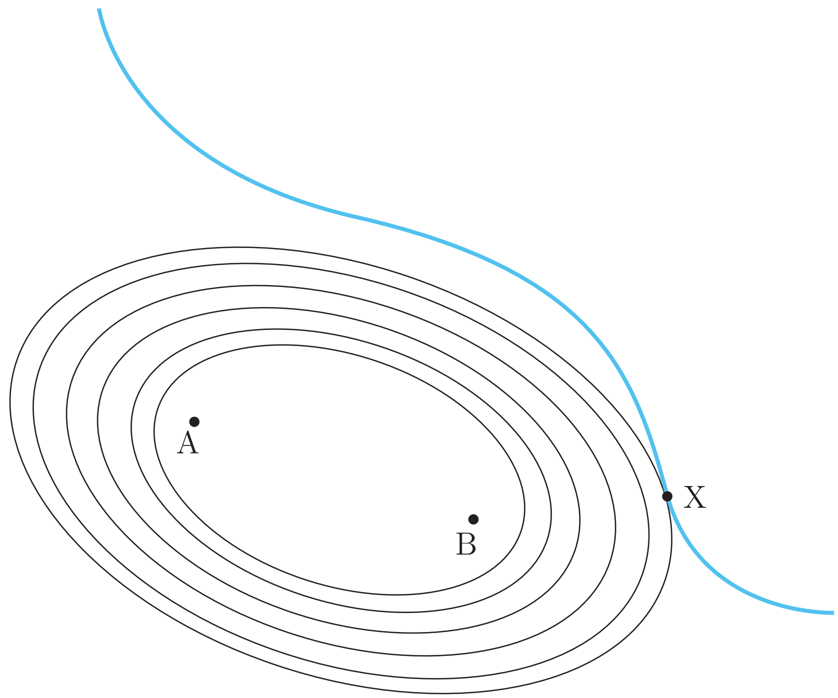

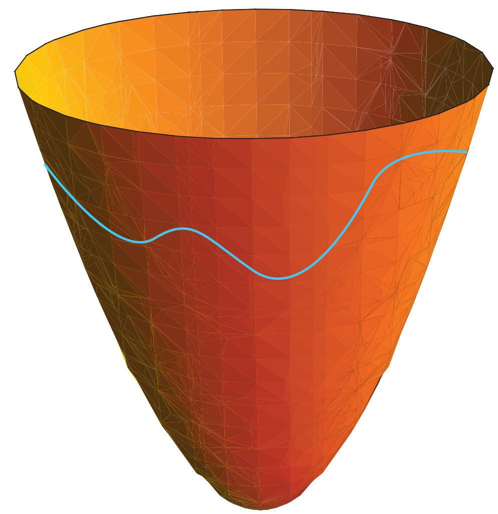

We have an elliptic paraboloid, $f(x, y)$. Imagine the path is the projection of a level curve of some function, $g(x, y) = c$ up onto the surface.
Then the river represents the level curve $g(x, y) = c$ and the ellipses are the level curves of $f(x, y)$.

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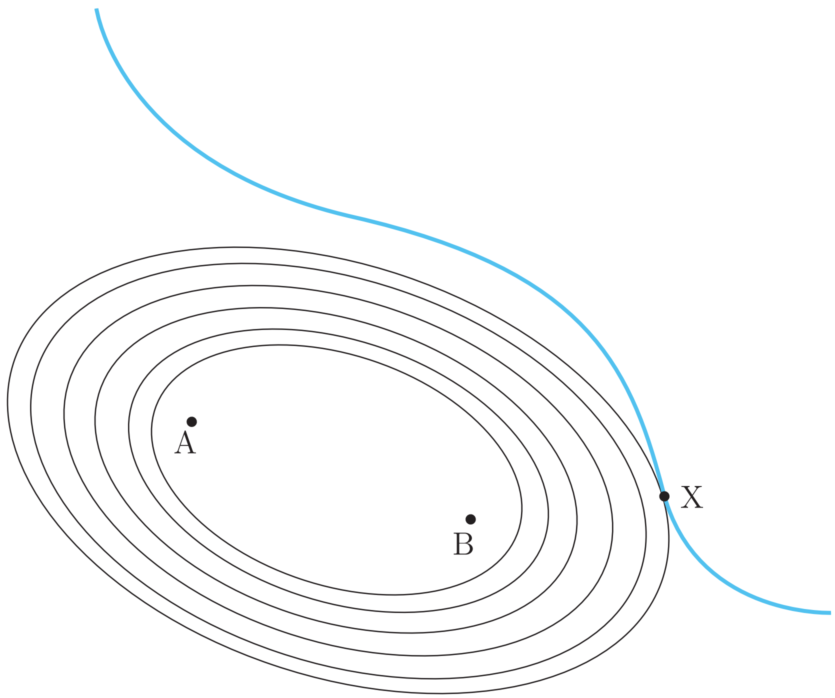

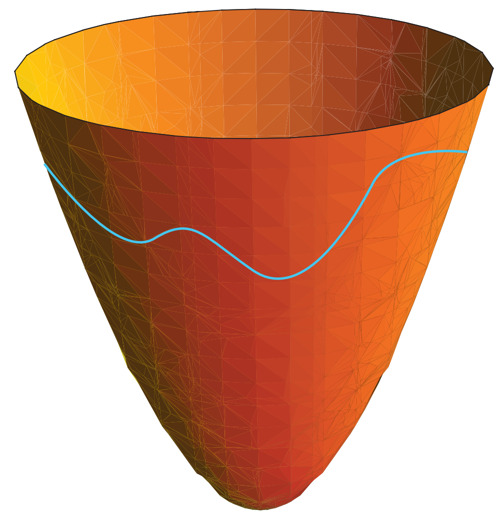

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The point of tangency between the level curves of f and the level curve $g(x,y) = c$ represents an extreme value of f restricted to the (projected) curve $g(x,y) = c$.

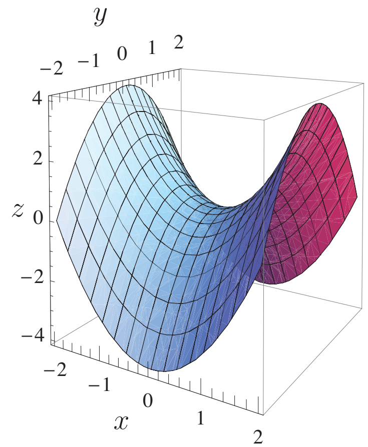
Lagrange Multipliers: Another form of constraint.

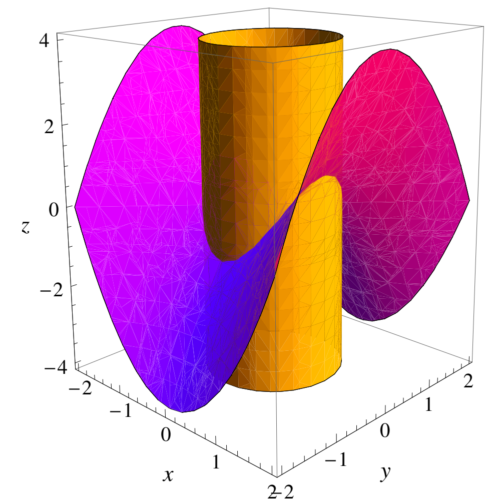
But first and example . . .

Example:

Let $f(x,y) = x^2 - y^2$ and let S be the circle of radius 1 about the origin: $g(x,y) = x^2 + y^2 = 1$.

Find the extrema of f constrained to S.





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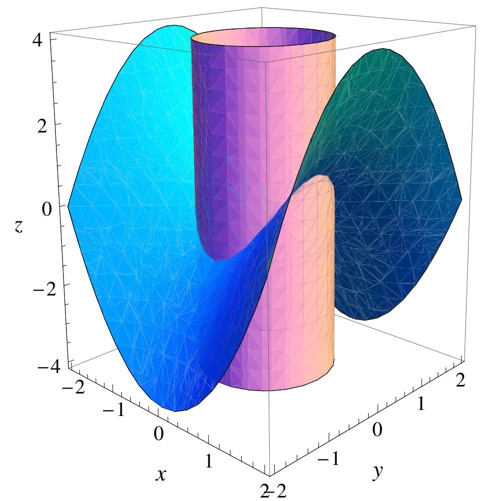
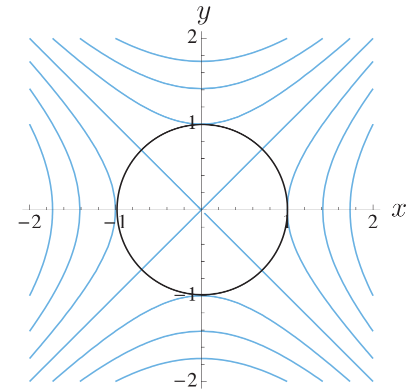
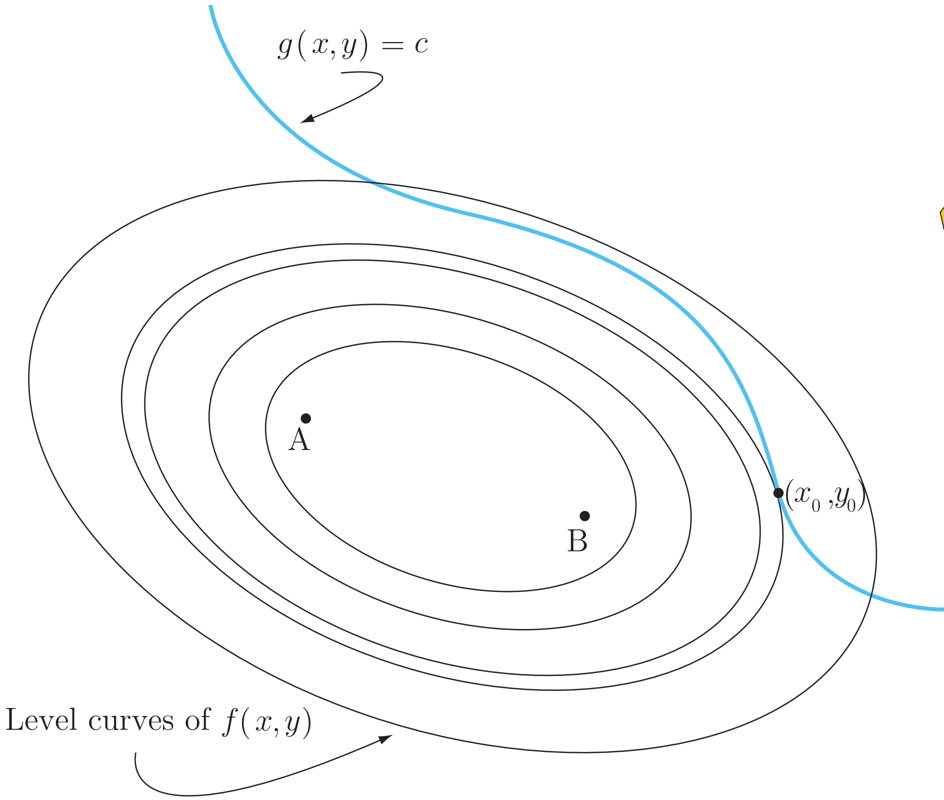


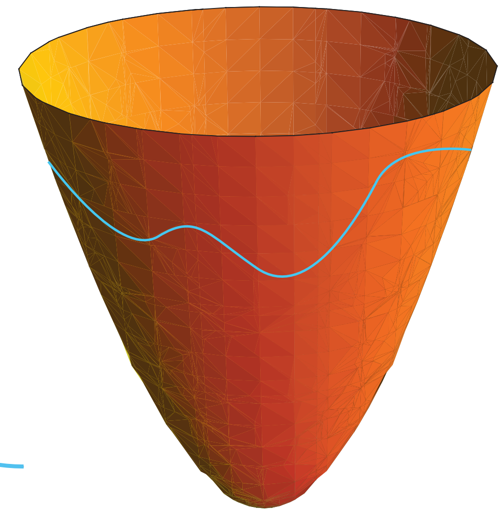


Geometrically we can find the solution by matching the level set of f to the level curve $x^2+y^2 = 1$. Therefore the extreme values of f constrained to S occur at $(0, \pm 1)$ and $(\pm 1, 0)$.

Lagrange Multipliers: Another form of constraint.

But why?





We want a general approach for minimizing or maximizing a function $f(x, y)$ when (x, y) is restricted to a level curve $g(x, y) = c$.
Suppose f and g are differentiable and since we are guaranteed a global minimum and a global maximum (the projection of $g(x, y) = c$ provides a boundary on which f is defined), let's call M the minimum value of f restricted to the level curve $g(x, y) = c$.

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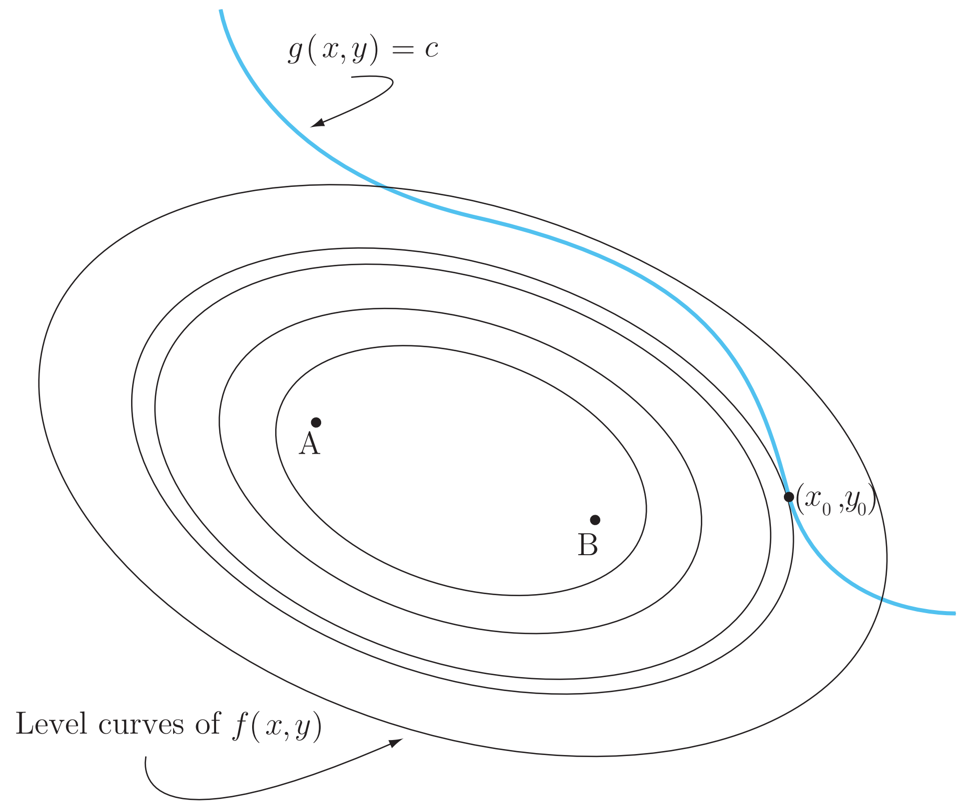
Then we consider the two curves, $f(x, y) = M$ and $g(x, y) = c$.



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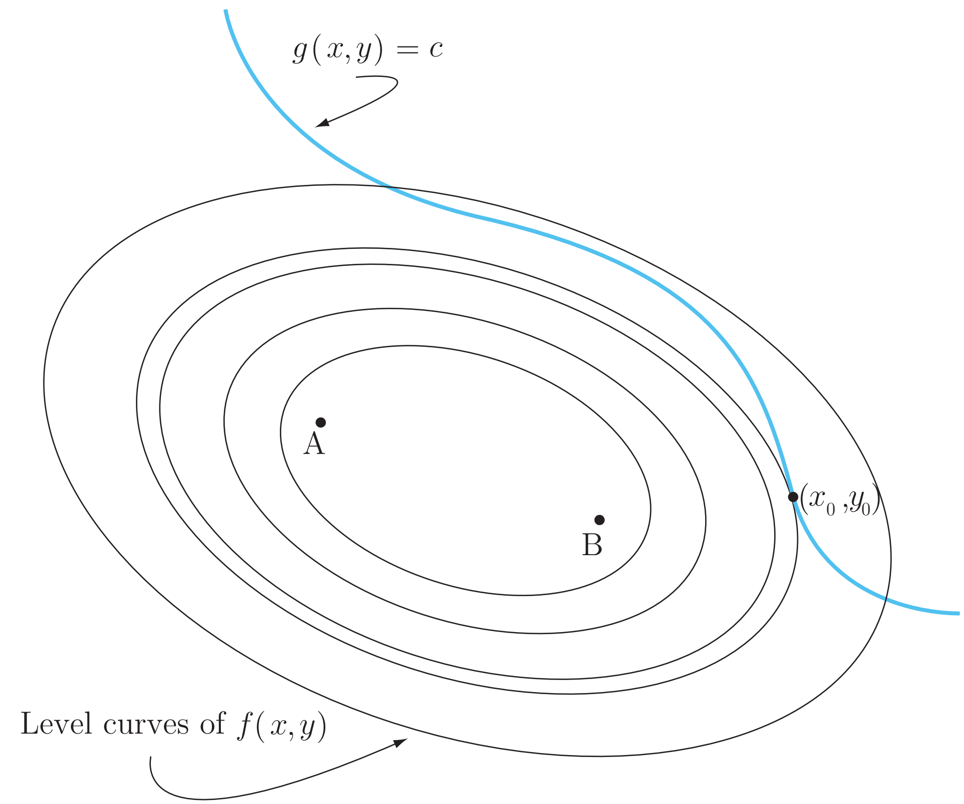
Then we consider the two curves, $f(x,y) = M$ and $g(x,y) = c$. The two curves intersect. Let's call the point of intersection (x_0, y_0). If we decrease M even slightly, then $g(x,y) = c$ does not intersect the level curve $f(x,y) = M$. Then at (x_0, y_0) we have a minimum, M, and the tangents to $f(x,y) = M$ and $g(x,y) = c$ are parallel (tangent curves).



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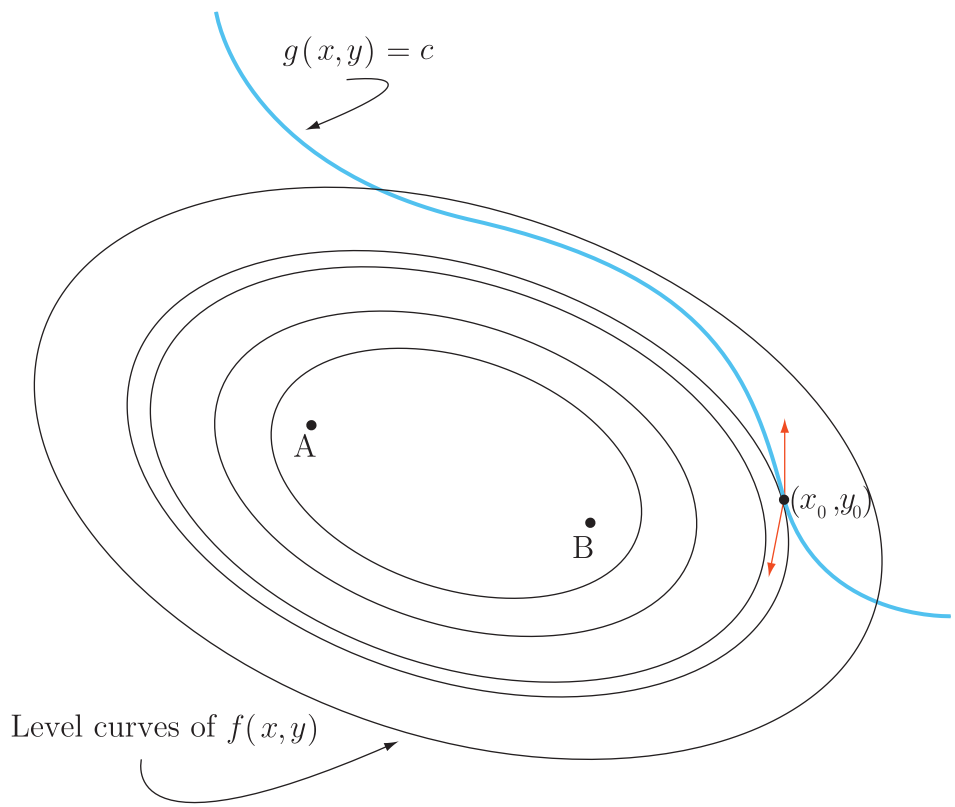

If they were not parallel then moving in one direction along $g(x, y) = c$ would cross lower level curves of f and moving in the opposite direction would cross higher level curves (so there would be a different minimum value).

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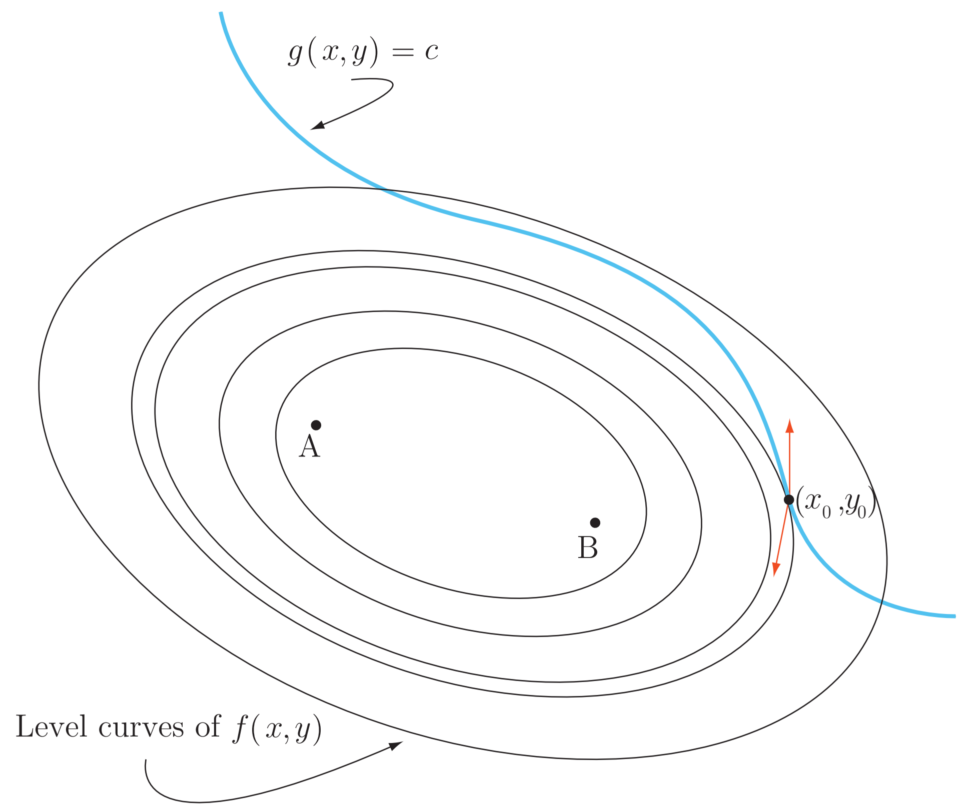

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If M is a local minimum then $\nabla f(x_0, y_0) = 0$.
If the tangents are parallel then so are the vectors normal to them so $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some constant $\lambda \in \mathbb{R}$.

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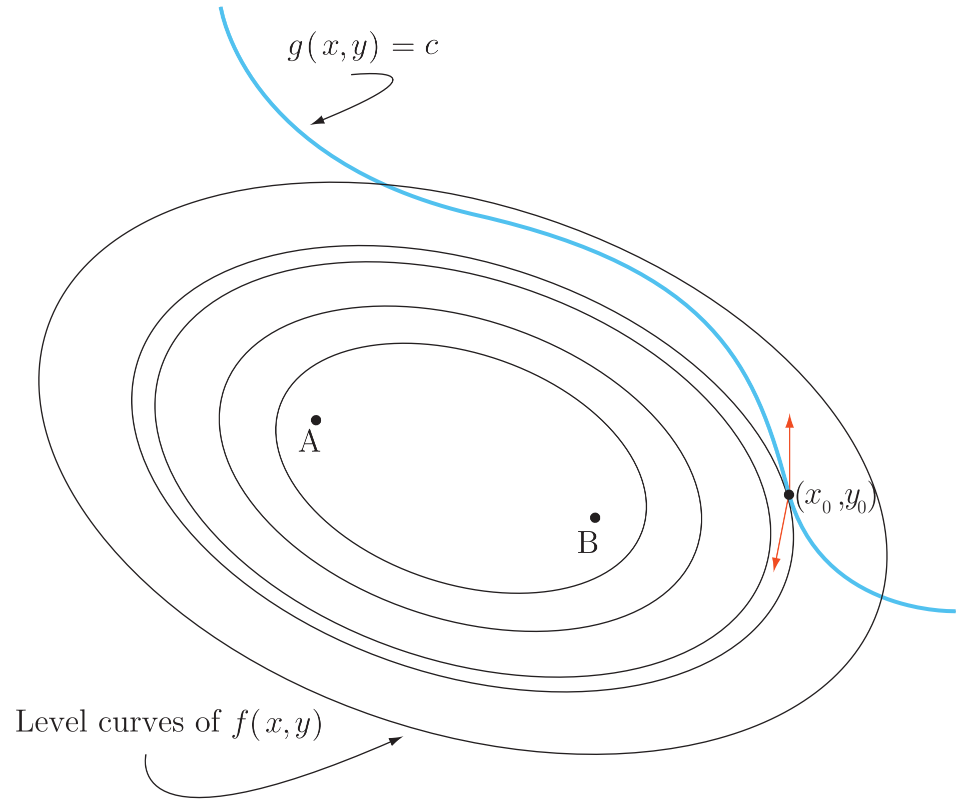
The case where $\nabla f(x_0, y_0) = 0$ is covered by $\lambda = 0$ so in order to determine the points of extrema it is a matter of solving a system of equations in three unknowns (x, y, λ).

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

$$g(x, y) = c$$



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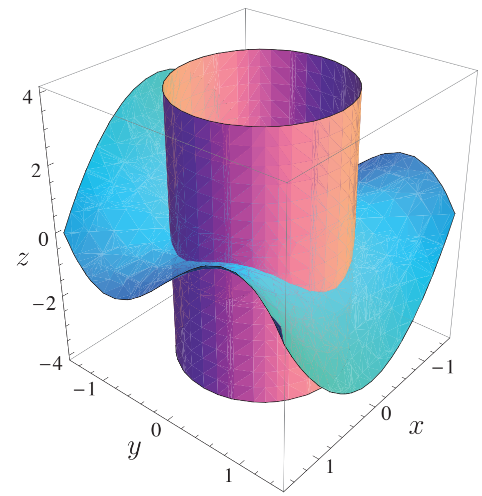

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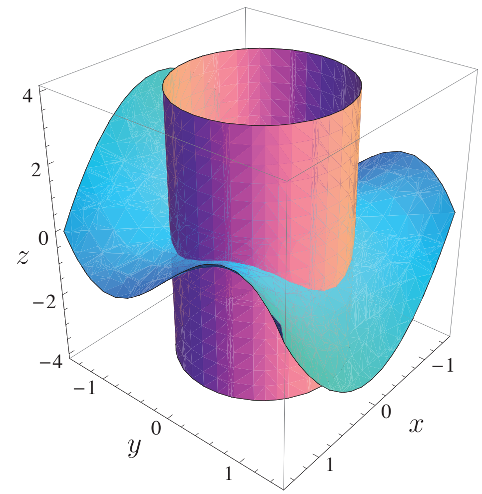

One Solution:

$$\begin{aligned} 2xy &= \lambda 2x \\ x^2 - 3y^2 &= \lambda 2y \\ x^2 + y^2 &= 1 \end{aligned} \quad \longrightarrow \quad \begin{aligned} &y = \lambda \text{ or } x = 0 \\ &x^2 = 5y^2 \end{aligned}$$

$$(x = 0) \to y = \pm 1 \text{ or } y = \pm\sqrt{\frac{1}{6}} \to \left(x = \pm\sqrt{\frac{5}{6}}\right)$$

It follows that the six critical points are $(0, \pm 1)$; $(\sqrt{5/6}, \pm\sqrt{1/6})$ and $(-\sqrt{5/6}, \pm\sqrt{1/6})$. The global maximum occurs at $f(0,-1) = 1$ and the global minimum occurs at $f(0,1) = -1$. $\square$

Lagrange Multipliers: Another form of constraint.

> **Theorem** (Lagrange Multipliers):
> Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable.
> Let $\vec{X}_0 \in \mathbb{R}^n$ with $g(\vec{X}_0) = c$ and let S be the level set for $g(\vec{X}) = c$ (the set of points $\vec{X} \in \mathbb{R}^n$ satisfying $g(\vec{X}) = c$). Assume $\nabla g \neq 0$.
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> If f restricted to S has a local maximum or minimum on S at $\vec{X}_0$ then there is a $\lambda \in \mathbb{R}$ such that $\nabla f(\vec{X}) = \lambda \nabla g(\vec{X})$.

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Proof Outline Recall in $\mathbb{R}^3$ we have seen the plane tangent to a surface, S, at (x_0, y_0) is orthogonal to $\nabla g(x_0, y_0)$. We can generalize this property to tangent spaces in $\mathbb{R}^n$. Observe that paths, $c(t)$ that lie in S will have tangent vectors $c'(t)$. But since $g(c(t)) = c$ it follows that $\frac{\mathrm{d}}{\mathrm{d}t} g(c(t)) = 0$. But assuming $c(0) = \vec{X}_0$ we also have $\frac{\mathrm{d}}{\mathrm{d}t} g(c(t))|_{t=0} = \nabla g(\vec{X}_0) \cdot c'(0)$.

Combining the two results we have $\nabla g(\vec{X}_0) \cdot c'(0) = 0$ and therefore $c'(0)$ is orthogonal to $\nabla g(\vec{X}_0)$.

If f restricted to S has a maximum at $\vec{X}_0$ then $f'(\vec{X}_0) = 0 = \frac{\mathrm{d}}{\mathrm{d}t} f(c(t))|_{t=0} = \nabla f(\vec{X}_0) \cdot c'(0)$.
So again we have $\nabla f(\vec{X}_0) \cdot c'(0) = 0$ and therefore $c'(0)$ is orthogonal to $\nabla f(\vec{X}_0)$.

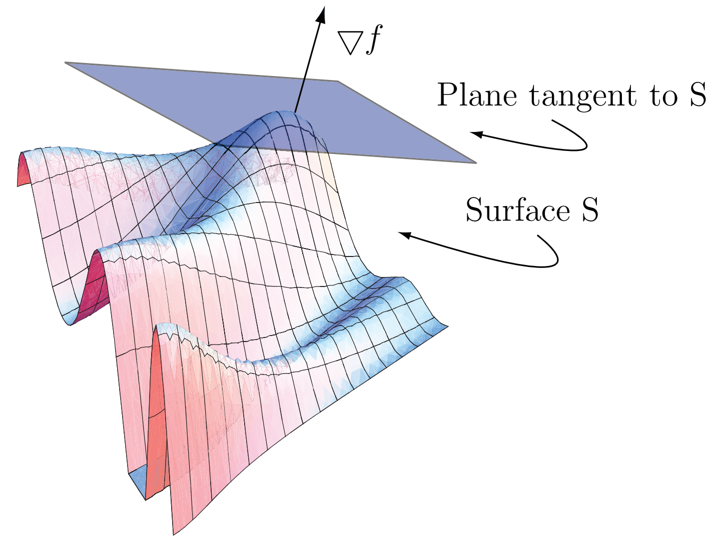
Since $\nabla f(\vec{X}_0)$ is perpendicular to the tangent of every curve in S it is perpendicular to the entire tangent space of S. Because the the space perpendicular to this tangent space is a line, $\nabla f(\vec{X}_0)$ and $\nabla g(\vec{X}_0)$ are parallel.

Lagrange Multipliers: Another form of constraint.

A direct consequence of the previous theorem is:

Theorem

If $f(\vec{X})$ when constrained to a surface, S, has a maximum or minimum at $\vec{X}_0$, then $\nabla f(\vec{X}_0)$ is perpendicular to S at $\vec{X}_0$.



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Yet Another Example:
Assume that among all rectangular boxes with fixed surface area of 10 square meters there is a box of largest possible volume. Find its dimensions.

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Solution:

If ℓ, w, and h are the dimensions of the box, then the function we wish to maximize is $V = \ell wh$ subject to the constraint that $2(\ell w + \ell h + wh) = 10$ or equivalently $\ell w + \ell h + wh = 5$.

Then we have the system of equations:

$$\begin{aligned} \ell w &= \lambda(\ell + w) \\ \ell h &= \lambda(\ell + h) \\ wh &= \lambda(w + h) \\ \ell w + \ell h + wh &= 5 \end{aligned}$$

We can see $\ell \neq 0$ since that would eliminate one dimension and we wouldn't have a box (moreover, $V = 0$). Likewise for w and h.

Solving for λ in the first two equations gives us $\dfrac{\ell w}{\ell + w} = \dfrac{\ell h}{\ell + h} \longrightarrow w\ell = h\ell \longrightarrow w = h$.

Similarly, from the second and third equations we get $\dfrac{wh}{w + h} = \dfrac{\ell h}{\ell + h} \longrightarrow wh = \ell h \longrightarrow w = \ell$.

Substituting into the constraint equations gives $w^2 + w^2 + w^2 = 5$ so $w = h = \ell = \sqrt{5/3}$.

Note that this proves the cube is the only possible candidate for the largest volume – but does not prove it is the box of greatest volume.