Dot Product

For $\vec{v} = (v_1, v_2, ..., v_n)$, $\vec{w} = (w_1, w_2, ..., w_n)$, $\vec{v} \cdot \vec{w} = \sum_{k=1}^{n} v_k w_k$

And equivalently $\vec{v} \cdot \vec{w} = ||\vec{v}|| \, ||\vec{w}|| \cos\theta$

The geometric definition is often used to determine the angle between two vectors:

$$\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \, ||\vec{w}||}\right)$$

. . . and to determine the work done in applying a force in the direction an object moves.
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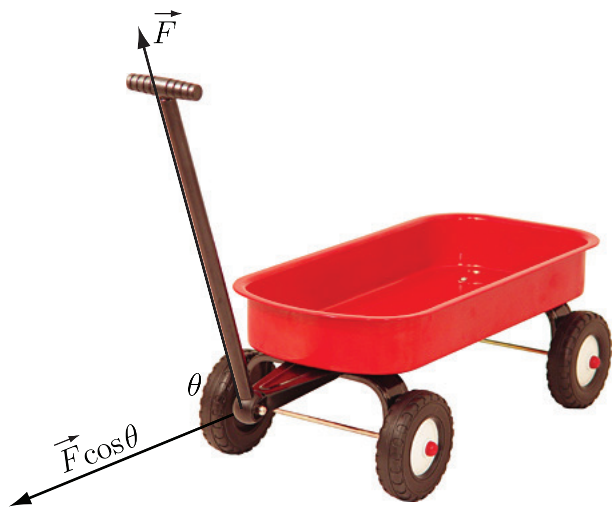

A Little History . . .

William Rowan Hamilton
(1805 – 1865)
Dublin

Complex Numbers $(a + bi), i = \sqrt{-1}$

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$$\begin{aligned}(a+bi)(c+di)& \\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

Complex Numbers $(a + bi)$

$$(a + bi)(c + di)$$
$$= (ac - bd) + (ad + bc)i$$

Hamilton: $\mathbb{C} = \mathbb{R}^2$

so $a + bi \Rightarrow (a, b)$

and $(a + bi)(c + di) \Rightarrow (a, b)(c, d) = (ac - bd, ad + bc)$

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Good question.

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- and . . .

Unfortunately . . .

Creating a rule for multiplication in $\mathbb{R}^3$ that retained the properties and consequences of multiplication found in $\mathbb{R}$ and $\mathbb{R}^2$ proved elusive.

Eventually (16 years after he began his pursuit), Hamilton had an epiphany . . .

Quaternions

If $\mathbb{R}^3$ won't comply, why not consider $\mathbb{R}^4$?

Hamilton described numbers of the form $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ where a was called the real or scalar part and $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ the vector or imaginary part.

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His epiphany?

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

. . . of course.

Quaternions

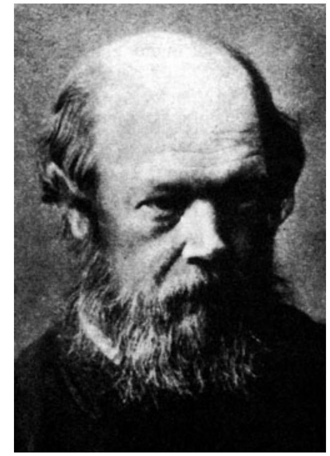
Hamilton's rules for quaternion multiplication are involved:

- $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$
- $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$
- $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$
- $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$

. . and while not commutative, *skew symmetry* was apparently close enough.

The set $\mathbb{R}^4$ with Hamilton's quaternion multiplication is usually denoted $\mathbb{H}$.

As luck would have it quaternions proved too cumbersome for most
(which may explain why Gauss, who had discovered many of the same results in 1819, never published his observations) and it wasn't until one of Hamilton's students, Peter Tait, found himself playing with the numbers that vector multiplication found its way into the math books.



Tait considered the product of $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\vec{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$
(note the absence of the fourth dimension).
His results were

$$(\vec{v})(\vec{w}) = -(v_1w_1 + v_2w_2 + v_3w_3) + (v_2w_3 - v_3w_2)\mathbf{i} + (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}$$

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and the vector product: $\vec{v} \times \vec{w} = (v_2w_3 - v_3w_2)\mathbf{i} + (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}$

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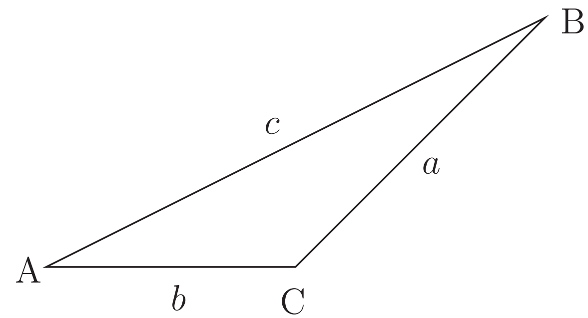
The Dot Product

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

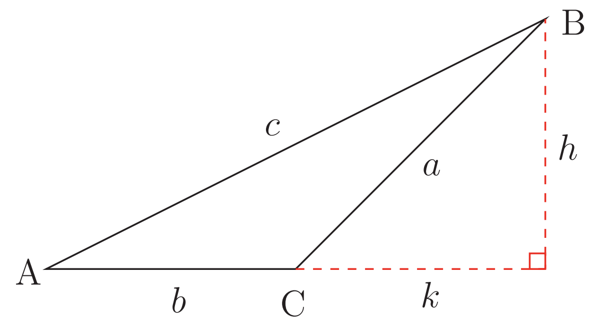


The Dot Product

Law of Cosines

Proof:

$$
\begin{aligned}
a^2 &= h^2 + k^2 \\
c^2 &= h^2 + (b+k)^2 \\
&= a^2 - k^2 + b^2 + 2bk + k^2 \\
&= a^2 + 2bk
\end{aligned}
$$



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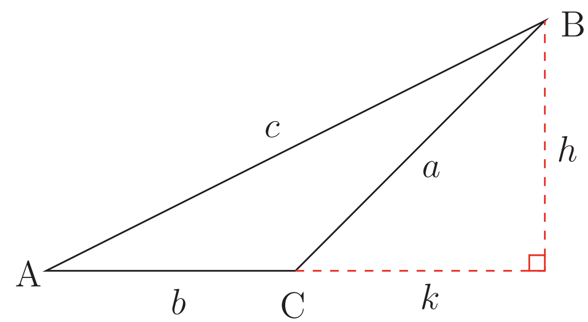

Now $\cos(180^\circ - C) = \dfrac{k}{a}$ so $k = a\cos(180^\circ - C)$.

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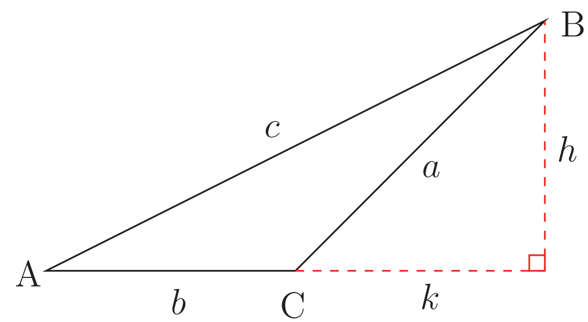

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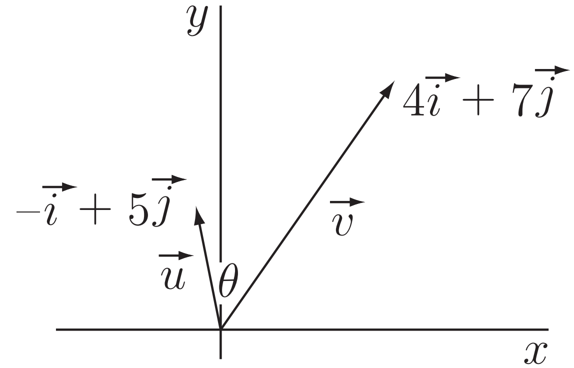
Vectors

Find the angle, θ between vectors $\vec{u}$ and $\vec{v}$.



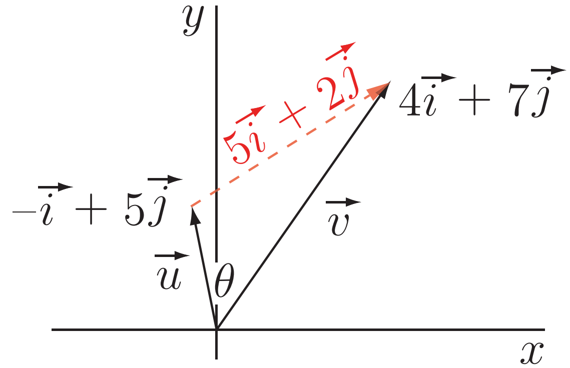
The Dot Product

Vectors

Find the angle, θ between vectors $\vec{u}$ and $\vec{v}$.

Solution:

Complete the triangle by finding the displacement vector, $\vec{v} - \vec{u} = 5\vec{i} + 2\vec{j}$.



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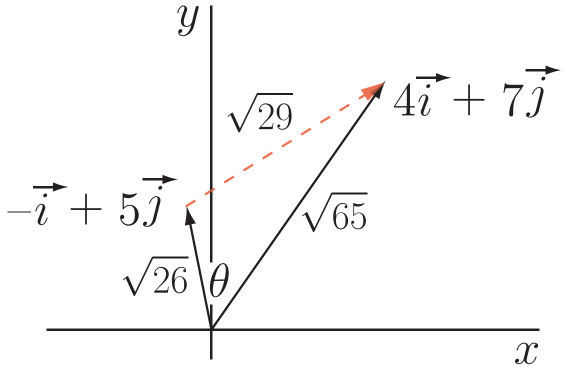

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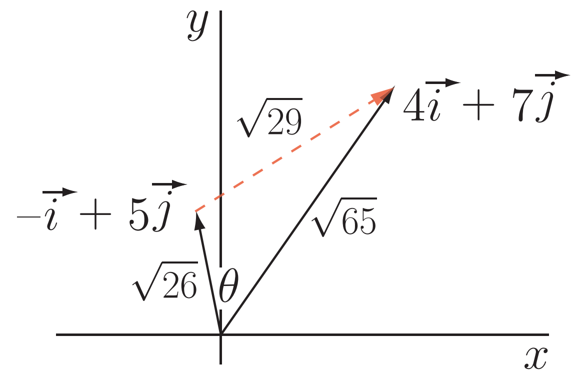

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Calculating the magnitudes of the vectors gives us the sides of the triangle. Then from the Law of Cosines we have

$$29 = 26 + 65 - 2\sqrt{26}\sqrt{65}\cos\theta$$

so $\cos\theta = \dfrac{29 - (26 + 65)}{-2\sqrt{26}\sqrt{65}} = \dfrac{26 + 65 - 29}{2\sqrt{26}\sqrt{65}}$

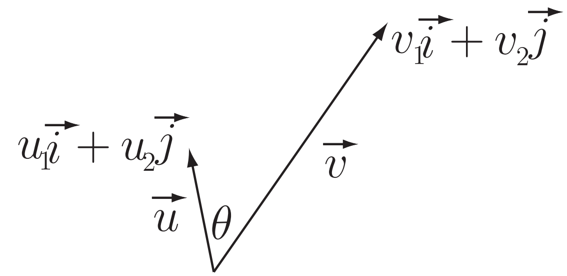
And $\theta = \arccos\left(\dfrac{31}{\sqrt{1690}}\right) \approx 41°$.

The Dot Product

Vectors

In general the Law of Cosines gives us some insight into the relationship between the coordinate form and trigonometric form of vectors.

Consider the vectors $\vec{u}$ and $\vec{v}$ with coordinate forms $\vec{u} = u_1\vec{i} + u_2\vec{j}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j}$.



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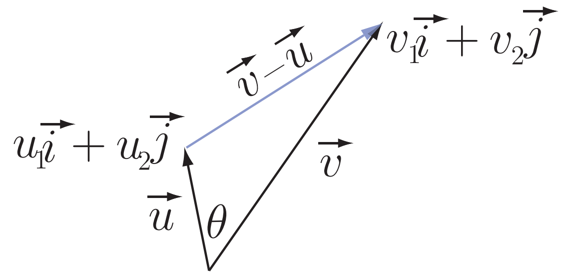

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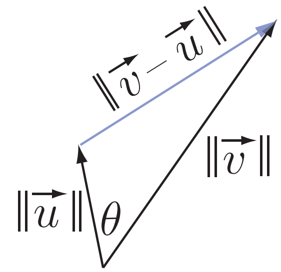

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Completing the triangle with displacement vector $\vec{v} - \vec{u}$, the magnitudes of the vectors give us the side lengths.

It follows that $||\vec{v} - \vec{u}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}||||\vec{v}|| \cos\theta$

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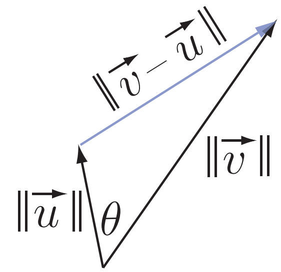

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Then substituting the coordinate forms, we have

$$\left(\sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}\right)^2 = \left(\sqrt{u_1^2 + u_2^2}\right)^2 + \left(\sqrt{v_1^2 + v_2^2}\right)^2 - 2||\vec{u}||||\vec{v}|| \cos\theta$$

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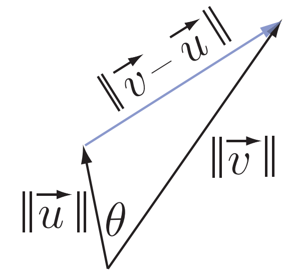

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Simplifying gives us

$$u_1v_1 + u_2v_2 = ||\vec{u}||||\vec{v}|| \cos\theta$$

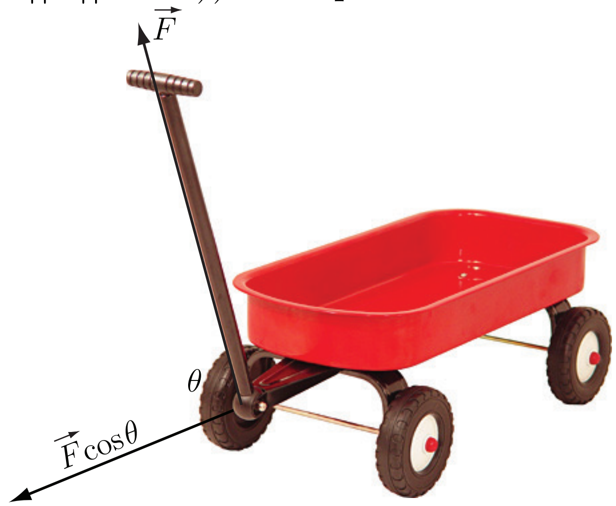
and consequently

$$\cos\theta = \frac{u_1v_1 + u_2v_2}{||\vec{u}||||\vec{v}||} \longrightarrow \theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||||\vec{v}||}\right)$$

From the equivalence $(u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2 = ||\vec{u}||||\vec{v}|| \cos\theta$

We often describe the dot product as the projection of $\vec{u}$ in the direction of $\vec{v}$.

That is, the component of $\vec{u}$ (of length $||\vec{u}|| \cos\theta$)) that points in the direction of $\vec{v}$.



Since work is defined as the product of force in the direction of motion...

$W = \vec{F} \cdot \vec{d}$

Some Consequences . . .

The geometric relationship $(u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2 = ||\vec{u}||||\vec{v}|| \cos\theta$ tells us that when $\vec{u}$ and $\vec{v}$ are perpendicular ($\theta = 90°$), the dot product is 0.

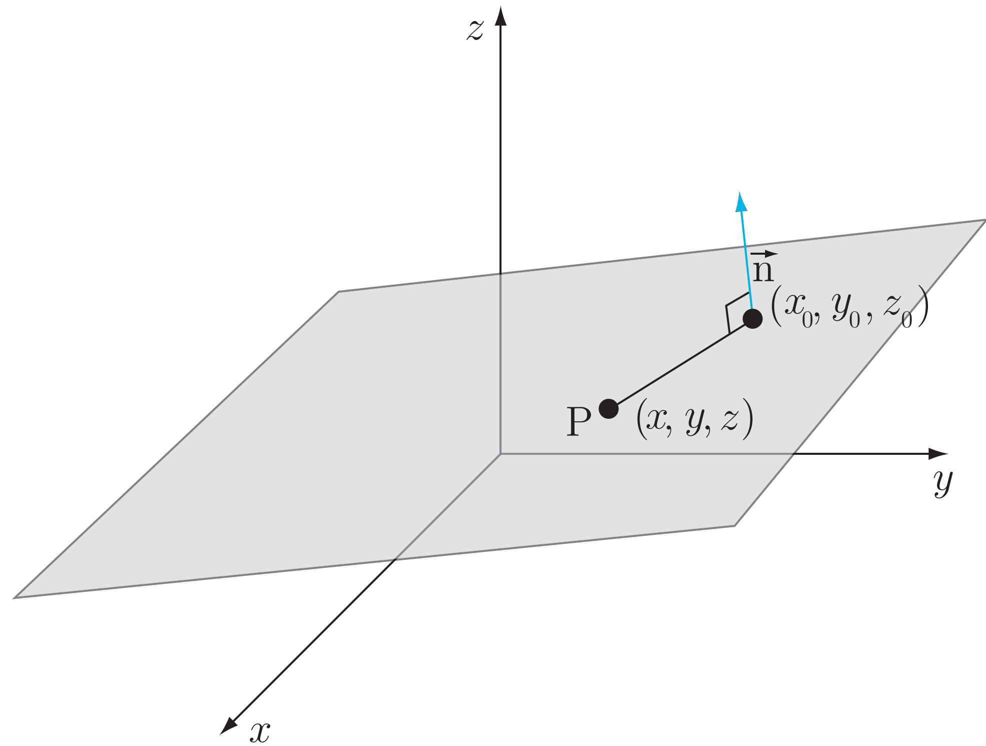
Conversely, assuming $\vec{u}$ and $\vec{v}$ are non-zero, a zero dot product tell us the two vectors are perpendicular.

Analytical model of a plane

A plane has the property that any two points lying in a plane define a line that must also lie entirely in the plane.

That means for a fixed point, (x_0, y_0, z_0) in a plane, the plane is composed of exactly those points in space, (x, y, z) that form lines with (x_0, y_0, z_0) entirely contained in the plane.

If $\vec{n} = n_1\vec{i} + n_2\vec{j} + n_3\vec{k}$ is a vector perpendicular to the plane, then we can describe the plane as the set of points, (x, y, z) whose displacement vectors through (x_0, y_0, z_0) are perpendicular to $\vec{n}$.



That is, those (x, y, z) whose displacement vector, $(x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$ has a zero dot product with $\vec{n}$:

$$((x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}) \cdot (n_1\vec{i} + n_2\vec{j} + n_3\vec{k}) =$$

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Dot Product

Matrix form

$$\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$