Notes

Chain Rule (Special case in $\mathbb{R}^3$, $h(t) = f(\mathbf{v}(t)) = f(x(t), y(t), z(t))$ where $\mathbf{v}(t) = (x(t), y(t), z(t))$):

$$\boxed{\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial f(x,y,z)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f(x,y,z)}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f(x,y,z)}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}}$$

Example: Find $\frac{\mathrm{d}h}{\mathrm{d}t}$ for $h(t) = f(\mathbf{v})$ where $f(x,y,z) = \frac{x}{y \ln z}$ and $\mathbf{v}(t) = \left(\sqrt{t}, \frac{1}{t}, t^2\right)$

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Solution:

$$\frac{\partial f}{\partial x} = \frac{1}{y \ln z} \longrightarrow \frac{1}{\frac{1}{t}\ln t^2} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2\sqrt{t}}$$

$$\frac{\partial f}{\partial y} = \frac{-x}{y^2 \ln z} \longrightarrow \frac{-\sqrt{t}}{\frac{1}{t^2}\ln t^2} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{t^2}$$

$$\frac{\partial f}{\partial z} = \frac{-x}{yz(\ln z)^2} \longrightarrow \frac{-\sqrt{t}}{\frac{1}{t}t^2(\ln t^2)^2} \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = 2t$$

$$\begin{aligned}
\frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{t}{\ln t^2}\cdot\frac{1}{2\sqrt{t}} + \frac{-t^{5/2}}{\ln t^2}\cdot\frac{-1}{t^2} + \frac{-1}{\sqrt{t}(\ln t^2)^2}\cdot 2t \\
&= \frac{\sqrt{t}}{2\ln t^2} + \frac{\sqrt{t}}{\ln t^2} - \frac{2\sqrt{t}}{(\ln t^2)^2} \\
&= \frac{3\sqrt{t}}{2\ln t^2} - \frac{2\sqrt{t}}{(\ln t^2)^2} \\
&= \frac{\sqrt{t}(3\ln t^2 - 4)}{2(\ln t^2)^2}
\end{aligned}$$

Chain Rule (Special case in $\mathbb{R}^3$):

Let $\mathbf{v} : \mathbb{R} \to \mathbb{R}^3$ be a differentiable path and $f : \mathbb{R}^3 \to \mathbb{R}$ a real valued function.

Suppose $h(t) = f(\mathbf{v}(t)) = f(x(t), y(t), z(t))$ where $\mathbf{v}(t) = (x(t), y(t), z(t))$.

Then

$$\begin{aligned}
\frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{\partial f(x,y,z)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f(x,y,z)}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f(x,y,z)}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\
&= \nabla f(\mathbf{v}(t)) \cdot \mathbf{v}'(t) \\
&= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \\ \frac{\mathrm{d}z}{\mathrm{d}t} \end{bmatrix}
\end{aligned}$$

Chain Rule (Special case in $\mathbb{R}^3$):

Let $\mathbf{v} : \mathbb{R} \to \mathbb{R}^3$ be a differentiable path and $f : \mathbb{R}^3 \to \mathbb{R}$ a real valued function. Suppose $h(t) = f(\mathbf{v}(t)) = f(x(t), y(t), z(t))$ where $\mathbf{v}(t) = (x(t), y(t), z(t))$. Then

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{\partial f(x,y,z)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f(x,y,z)}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f(x,y,z)}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= \nabla f(\mathbf{v}(t)) \cdot \mathbf{v}'(t)\end{aligned}$$

Where $\mathbf{v}'(t) = (x'(t), y'(t), z'(t))$

Proof:

$$h'(t_0) = \lim_{t \to t_0} \frac{h(t) - h(t_0)}{t - t_0}$$

$$\begin{aligned}\frac{h(t) - h(t_0)}{t - t_0} &= \frac{f(x(t), y(t), z(t)) - f(x(t_0), y(t_0), z(t_0))}{t - t_0} \\ &= \frac{f(x(t), y(t), z(t)) - f(x(t_0), y(t), z(t))}{t - t_0} \\ &\quad + \frac{f(x(t_0), y(t), z(t)) - f(x(t_0), y(t_0), z(t))}{t - t_0} \\ &\quad + \frac{f(x(t_0), y(t_0), z(t)) - f(x(t_0), y(t_0), z(t_0))}{t - t_0}\end{aligned}$$

$$
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\frac{h(t)-h(t_0)}{t-t_0} &= \frac{f(x(t),y(t),z(t)) - f(x(t_0),y(t_0),z(t_0))}{t-t_0} \\
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\end{aligned}
$$

From the Mean Value Theorem it follows that there is some $c \in (x, x_0)$ such that

$$
\frac{f(x,y,z) - f(x_0,y,z)}{x-x_0} = \frac{\partial f(c,y,z)}{\partial x} \longrightarrow f(x,y,z) - f(x_0,y,z) = \frac{\partial f(c,y,z)}{\partial x}(x-x_0)
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Then similarly for $c \in (x(t), x(t_0))$, $d \in (y(t), y(t_0))$ and $e \in (z(t), z(t_0))$

It follows that

$$
\frac{h(t)-h(t_0)}{t-t_0} = \left(\frac{\partial f(c,y(t),z(t))}{\partial x}\right)\frac{x(t)-x(t_0)}{t-t_0} + \left(\frac{\partial f(x(t_0),d,z(t))}{\partial y}\right)\frac{y(t)-y(t_0)}{t-t_0} + \left(\frac{\partial f(x(t_0),y(t_0),e)}{\partial z}\right)\frac{z(t)-z(t_0)}{t-t_0}
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$$\frac{h(t)-h(t_0)}{t-t_0} = \left(\frac{\partial f(c,y(t),z(t))}{\partial x}\right)\frac{x(t)-x(t_0)}{t-t_0} + \left(\frac{\partial f(x(t_0),d,z(t))}{\partial y}\right)\frac{y(t)-y(t_0)}{t-t_0} + \left(\frac{\partial f(x(t_0),y(t_0),e)}{\partial z}\right)\frac{z(t)-z(t_0)}{t-t_0}$$

$$\text{As } t \to t_0, \quad = \left(\frac{\partial f(c,y(t),z(t))}{\partial x}\right)\frac{\mathrm{d}x}{\mathrm{d}t} + \left(\frac{\partial f(x(t_0),d,z(t))}{\partial y}\right)\frac{\mathrm{d}y}{\mathrm{d}t} + \left(\frac{\partial f(x(t_0),y(t_0),e)}{\partial z}\right)\frac{\mathrm{d}z}{\mathrm{d}t}$$

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\begin{aligned}
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&= \frac{f(x(t),y(t),z(t))-f(x(t_0),y(t),z(t))}{t-t_0} \\
&\quad + \frac{f(x(t_0),y(t),z(t))-f(x(t_0),y(t_0),z(t))}{t-t_0} \\
&\quad + \frac{f(x(t_0),y(t_0),z(t))-f(x(t_0),y(t_0),z(t_0))}{t-t_0}
\end{aligned}
$$

From the Mean Value Theorem it follows that there is some $c \in (x, x_0)$ such that

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\frac{f(x,y,z)-f(x_0,y,z)}{x-x_0} = \frac{\partial f(c,y,z)}{\partial x} \longrightarrow f(x,y,z)-f(x_0,y,z) = \frac{\partial f(c,y,z)}{\partial x}(x-x_0)
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Then similarly for $c \in (x(t), x(t_0))$, $d \in (y(t), y(t_0))$ and $e \in (z(t), z(t_0))$

It follows that

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\frac{h(t)-h(t_0)}{t-t_0} = \left(\frac{\partial f(c,y(t),z(t))}{\partial x}\right)\frac{x(t)-x(t_0)}{t-t_0} + \left(\frac{\partial f(x(t_0),d,z(t))}{\partial y}\right)\frac{y(t)-y(t_0)}{t-t_0} + \left(\frac{\partial f(x(t_0),y(t_0),e)}{\partial z}\right)\frac{z(t)-z(t_0)}{t-t_0}
$$

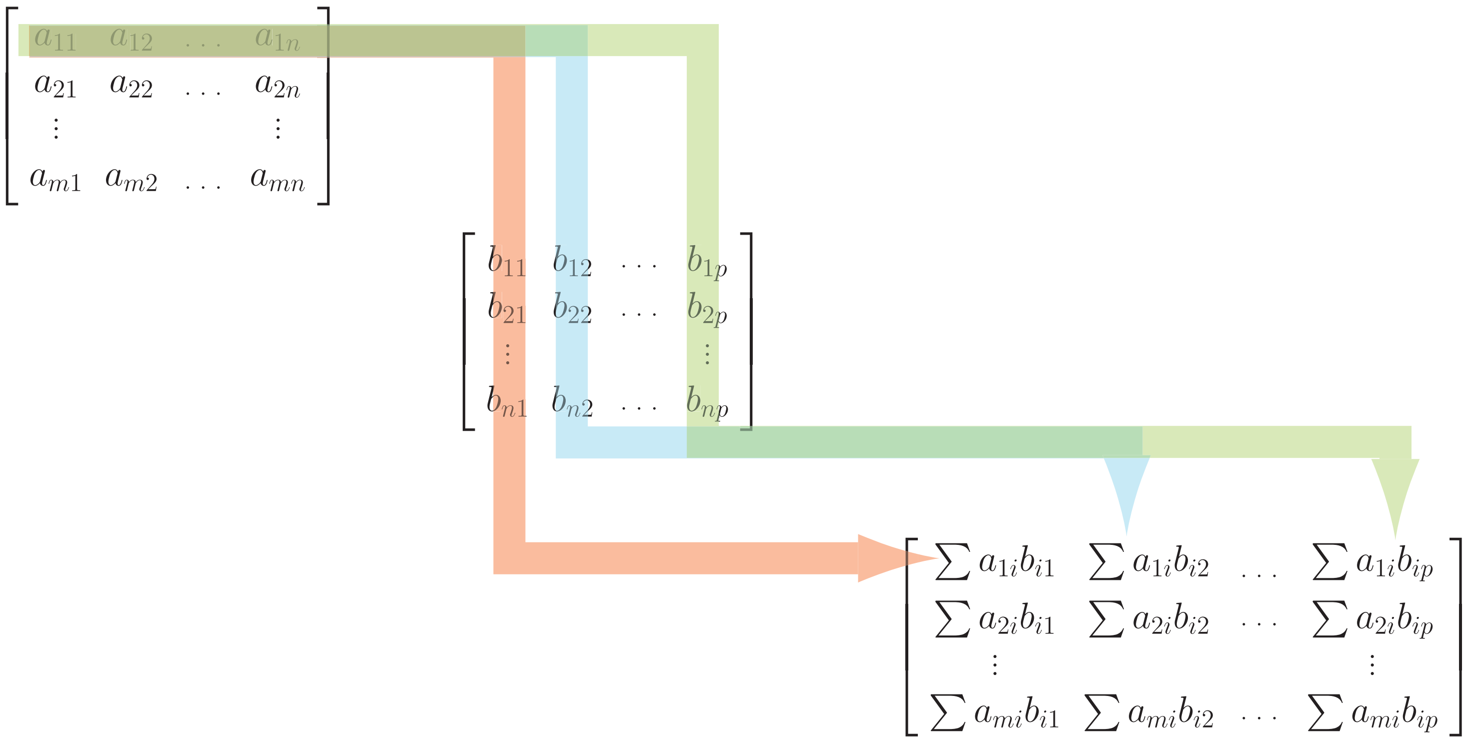
$$
\text{As } t \to t_0, \quad = \left(\frac{\partial f(c,y(t),z(t))}{\partial x}\right)\frac{\mathrm{d}x}{\mathrm{d}t} + \left(\frac{\partial f(x(t_0),d,z(t))}{\partial y}\right)\frac{\mathrm{d}y}{\mathrm{d}t} + \left(\frac{\partial f(x(t_0),y(t_0),e)}{\partial z}\right)\frac{\mathrm{d}z}{\mathrm{d}t}
$$

And as $t \to t_0$, we have $c \to x(t_o), d \to y(t_0), e \to z(t_0)$ we have

$$
\boxed{\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial f(x,y,z)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f(x,y,z)}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f(x,y,z)}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}}
$$

Appendix: Matrix Multiplication

$$\underset{m \times n \text{ matrix}}{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}} \underset{n \times p \text{ matrix}}{\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}} = \underset{m \times p \text{ matrix}}{\begin{bmatrix} \sum a_{1i}b_{i1} & \sum a_{1i}b_{i2} & \dots & \sum a_{1i}b_{ip} \\ \sum a_{2i}b_{i1} & \sum a_{2i}b_{i2} & \dots & \sum a_{2i}b_{ip} \\ \vdots & & & \vdots \\ \sum a_{mi}b_{i1} & \sum a_{mi}b_{i2} & \dots & \sum a_{mi}b_{ip} \end{bmatrix}}$$



Appendix: Matrix Multiplication

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 7 & 3 \\ 2 & 6 & 1 \\ 8 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 33 & 14 \\ 56 & 69 & 43 \end{bmatrix}$$

Chain Rule (Another special case in $\mathbb{R}^3$):

Let $\mathbf{w} : \mathbb{R}^2 \to \mathbb{R}^3$ be a differentiable path and $f : \mathbb{R}^3 \to \mathbb{R}$ a real valued function.

Suppose $h(u, v) = f(\mathbf{w}(u, v)) = f(x(u, v), y(u, v), z(u, v))$ where $\mathbf{w}(u, v) = (x(u, v), y(u, v), z(u, v))$.

Then

$$\frac{\partial h}{\partial u} = \frac{\partial f(x,y,z)}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f(x,y,z)}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f(x,y,z)}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial h}{\partial v} = \frac{\partial f(x,y,z)}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f(x,y,z)}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f(x,y,z)}{\partial z}\frac{\partial z}{\partial v}$$

$$\begin{bmatrix} \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$$

Example:

If $f(x, y, z) = \ln(x^2 + y^2 + z^2)$ and $g(u, v) = (u^2 + v^2, u^3, \sqrt{v})$, find $\mathbf{D}(f \circ g)(0, 1)$.

Example:

If $f(x,y,z) = \ln(x^2+y^2+z^2)$ and $g(u,v) = (u^2+v^2, u^3, \sqrt{v})$, find $\mathbf{D}(f \circ g)(0,1)$.

Solution:

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2+z^2} \qquad \frac{\partial x}{\partial u} = 2u \qquad \frac{\partial x}{\partial v} = 2v$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2+z^2} \qquad \frac{\partial y}{\partial u} = 3u^2 \qquad \frac{\partial y}{\partial v} = 0$$

$$\frac{\partial f}{\partial z} = \frac{2z}{x^2+y^2+z^2} \qquad \frac{\partial z}{\partial u} = 0 \qquad \frac{\partial z}{\partial v} = \frac{1}{2\sqrt{v}}$$

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$$\begin{bmatrix} \frac{2x}{x^2+y^2+z^2} & \frac{2y}{x^2+y^2+z^2} & \frac{2z}{x^2+y^2+z^2} \end{bmatrix} \begin{bmatrix} 2u & 2v \\ 3u^2 & 0 \\ 0 & \frac{1}{2\sqrt{v}} \end{bmatrix}$$

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If $f(x,y,z) = \ln(x^2+y^2+z^2)$ and $g(u,v) = (u^2+v^2, u^3, \sqrt{v})$, find $\mathbf{D}(f \circ g)(0,1)$.

Solution:

$$\left[\frac{2x}{x^2+y^2+z^2} \quad \frac{2y}{x^2+y^2+z^2} \quad \frac{2z}{x^2+y^2+z^2}\right] \begin{bmatrix} 2u & 2v \\ 3u^2 & 0 \\ 0 & \frac{1}{2\sqrt{v}} \end{bmatrix}$$

$$\left[\frac{2(u^2+v^2)}{(u^2+v^2)^2+u^6+v} \quad \frac{2u^3}{(u^2+v^2)^2+u^6+v} \quad \frac{2\sqrt{v}}{(u^2+v^2)^2+u^6+v}\right] \begin{bmatrix} 2u & 2v \\ 3u^2 & 0 \\ 0 & \frac{1}{2\sqrt{v}} \end{bmatrix}\Bigg|_{(0,1)}$$

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If $f(x,y,z) = \ln(x^2+y^2+z^2)$ and $g(u,v) = (u^2+v^2, u^3, \sqrt{v})$, find $\mathbf{D}(f \circ g)(0,1)$.

Solution:

$$\begin{bmatrix} \dfrac{2x}{x^2+y^2+z^2} & \dfrac{2y}{x^2+y^2+z^2} & \dfrac{2z}{x^2+y^2+z^2} \end{bmatrix} \begin{bmatrix} 2u & 2v \\ 3u^2 & 0 \\ 0 & \dfrac{1}{2\sqrt{v}} \end{bmatrix}$$

$$\left.\begin{bmatrix} \dfrac{2(u^2+v^2)}{(u^2+v^2)^2+u^6+v} & \dfrac{2u^3}{(u^2+v^2)^2+u^6+v} & \dfrac{2\sqrt{v}}{(u^2+v^2)^2+u^6+v} \end{bmatrix} \begin{bmatrix} 2u & 2v \\ 3u^2 & 0 \\ 0 & \dfrac{1}{2\sqrt{v}} \end{bmatrix}\right|_{(0,1)}$$

$$= [1 \quad 0 \quad 1] \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = [0 \quad 2.5]$$

Example:

Suppose a duck is swimming in a circle $x = \cos t$, $y = \sin t$ and that the water temperature is given by $W = x^2 e^y - xy^3$. Find $\mathrm{d}W/\mathrm{d}t$, the rate of change in the temperature the duck might feel as it swims.

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Solution:

$$\begin{aligned}
\frac{\mathrm{d}W}{\mathrm{d}t} &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{bmatrix} \\
&= \begin{bmatrix} 2xe^y - y^3 & x^2 e^y - 3xy^2 \end{bmatrix} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \\
&= (2xe^y - y^3)(-\sin t) + (x^2 e^y - 3xy^2)(\cos t) \\
&= -2 \sin t \cos t e^{\sin t} + \sin^4 t + \cos^3 t e^{\sin t} - 3 \cos^2 t \sin^2 t
\end{aligned}$$

Directional Derivative (again)

In the case where $f : \mathbb{R}^3 \to \mathbb{R}$ we let $\vec{X} = (x, y, z)$ and claim that

$$\begin{aligned} f_{\vec{\mathrm{u}}}(x, y, z) &= \nabla f(x, y, z) \cdot \vec{\mathrm{u}} \\ &= \frac{\partial f(x, y, z)}{\partial x} u_1 + \frac{\partial f(x, y, z)}{\partial y} u_2 + \frac{\partial f(x, y, z)}{\partial z} u_3 \end{aligned}$$

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Proof:

Recall The Directional Derivative of f at $\vec{X}$ with respect to unit vector $\vec{\mathrm{u}}$ is defined by

$$f_{\vec{\mathrm{u}}}(\vec{X}) = f'(\vec{X}; \vec{\mathrm{u}}) = \lim_{h \to 0} \frac{f(\vec{X} + h\vec{\mathrm{u}}) - f(\vec{X})}{h}$$

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Equivalently we can write $f_{\vec{\mathrm{u}}}(\vec{X}) = \frac{\mathrm{d}}{\mathrm{d}t} f(\vec{X} + t\vec{\mathrm{u}}) \Big|_{t=0}$

Let $\mathbf{v}(t) = \vec{X} + t\vec{\mathrm{u}}$ then $f(\vec{X} + t\vec{\mathrm{u}}) = f(\mathbf{v}(t))$

From the chain rule $\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{v}(t)) = \nabla f(\mathbf{v}(t)) \cdot \mathbf{v}'(t)$.

Since $\mathbf{v}(0) = \vec{X}$ and $\mathbf{v}'(0) = \vec{\mathrm{u}}$ we have $f_{\vec{\mathrm{u}}}(x, y, z) = \frac{\partial f(x, y, z)}{\partial x} u_1 + \frac{\partial f(x, y, z)}{\partial y} u_2 + \frac{\partial f(x, y, z)}{\partial z} u_3$ $\square$

Review:

Let $z = x^2y^3 + y^2 - 1$. Find the equation of the plane tangent to this surface at the point $(-1, 1, 1)$.

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Solution: We know that the gradient of $f(x, y, z)$ is perpendicular to the level surfaces of f so we write: $f(x, y, z) = x^2y^3 + y^2 - 1 - z$ and then find ∇f.

$$\begin{aligned}\nabla f &= 2xy^3\hat{\imath} + (3x^2y^2 + 2y)\hat{\jmath} - \hat{k}\Big|_{(-1,1,1)} \\ &= -2\hat{\imath} + 5\hat{\jmath} - \hat{k}\end{aligned}$$

Then the tangent plane at $(-1, 1, 1)$ is given by $-2(x + 1) + 5(y - 1) - (z - 1) = 0$