Differential Equations Name:________________

Approximate Solutions Through Taylor Polynomials.

Suppose we wish to solve the second order differential equation below, subject to the given initial conditions.

$$y'' + xy = 0 \qquad y(0) = 1, y'(0) = 0$$

It appears that seprating variables will not produce a solution and if we assume that no solution exists or is available through known methods, then the following approximation may suffice, so long as the values to be used are close to the center of the series. A theorem exists to test for radius of convergence but we will not be concerned with this for now.
Recall that a Taylor series centered at 0 has the form

$$f(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots \qquad \text{Where } C_n = \frac{f^{(n)}(0)}{n!}$$

We begin to approximate y as follows:

$y(0) = 1$ (Given) so $C_0 = 1$

$y'(0) = 0$ (Given) so $C_1 = 0$

$y''(0) = 0$ ($y'' = -xy$ from diff. eq.) so $C_2 = 0$

$y'''(0) = -1$ ($y''' = -y - xy'$ from above) so $C_3 = \frac{-1}{3!} = \frac{-1}{6}$

and so on. Then the third degree approximation of y is given by

$$y = 1 - \frac{1}{6}x^3$$

□

Notice that this does not explicitly satisfy the differential equation (try it!)

Your turn: Find the first four non-zero terms of the Taylor Polynomial approximation to the solution of the differential equation below.

$$y'' + xy' + y = 0 \qquad y(0) = 1, y'(0) = 0$$

Ans: $P_6(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6$

More exercises:
Find the first four non-zero terms of the Taylor Polynomial approximation about $x = 0$ subject to the given condition.

(a) $y' = x^2 + y^2, \quad y(0) = 1$

Ans:
$P_3(x) = 1 + x + x^2 + \frac{4}{3}x^3$

(b) $y' = 1 + xy^2, \quad y(0) = 0$

Ans:
$P_{10}(x) = x + \frac{1}{4}x^4 + \frac{1}{14}x^7 + \frac{23}{1120}x^{10}$

(c) $y'' = xy' \quad y(0) = 1,\ y'(0) = 1$

Ans:
$P_5(x) = 1 + x + \frac{1}{3!}x^3 + \frac{3}{5!}x^5$