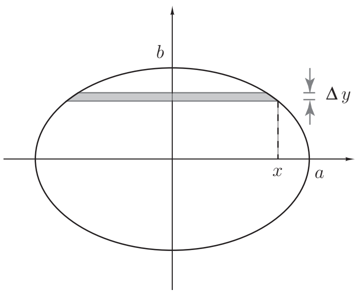
Solids of Revolution — Name:________________________

Show all relevant work!

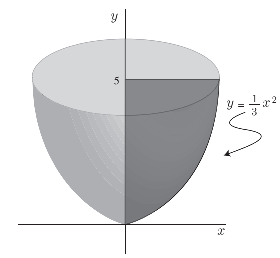
1. Use calculus to find the area of an ellipse with the formula $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



Solution: $x = a\sqrt{1-\frac{y^2}{b^2}} = \frac{a}{b}\sqrt{b^2-y^2}$ so the width of the rectangular strip is $2\frac{a}{b}\sqrt{b^2-y^2}$.
The area of one strip of height Δy is therefore $2\frac{a}{b}\sqrt{b^2-y^2}\,\Delta y$.

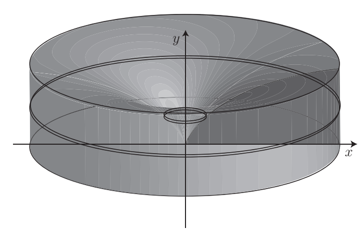
$$\text{Area} = \int_{-b}^{b} 2\frac{a}{b}\sqrt{b^2-y^2}\,\mathrm{d}y = 4\frac{a}{b}\int_0^b \sqrt{b^2-y^2}\,\mathrm{d}y$$

2. Find the volume of the solid generated by revolving the area bounded by $y=\frac{1}{3}x^2$ between $y=0$, $y=5$, and the y–axis about the y–axis.



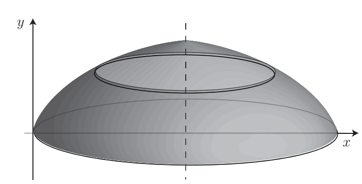
Solution: $x=\sqrt{3y}$ so $V = \displaystyle\int_0^5 \pi(\sqrt{3y})^2\mathrm{d}y = \frac{3\pi}{2}y^2\Big|_0^5 = \frac{75}{2}\pi$

3. Find the volume of the solid generated by revolving the area bounded by $y=\sqrt{x}$ between $x=0$, $x=4$, and the x–axis about the y–axis.



Solution: The outer radius is $x=4$ while the inner radius is $x=y^2$.
Then the area of a washer slice is given by $\Delta A = \pi 4^2 - \pi(y^2)^2$.
When $x=4$, $y=2$ so $V=\int_0^2 \pi(16-y^4)\mathrm{d}y = \pi(16y-\frac{1}{5}y^5)\Big|_0^2 = \frac{128}{5}\pi$

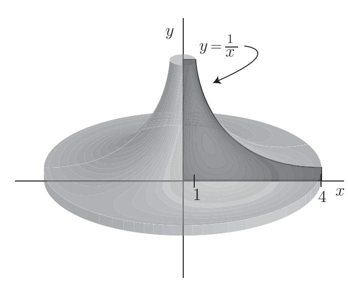
4. Repeat #3 for the volume of the solid generated by revolving the area bounded by $y=\sqrt{x}$ between $x=0$, $x=4$, and the x–axis about the axis $x=4$.



Solution: In this case the radius of each disc is $r=4-y^2$.
Then $\Delta A = \pi(4-y^2)^2$ so $V = \displaystyle\int_0^2 \pi(4-y^2)^2\mathrm{d}y = \pi\int_0^2 16-8y^2+y^4\,\mathrm{d}y$

$$= \pi(16y-\frac{8}{3}y^3+\frac{1}{5}y^5)\Big|_0^2 = \frac{256}{15}\pi$$

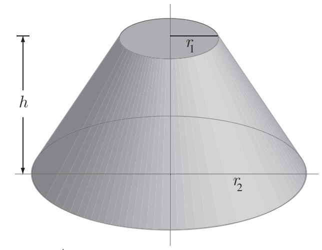
5. Find the volume of the solid shown to the right.



Solution: Since $x=\frac{1}{y}$ it follows that the area of each disc is $\Delta A = \pi(\frac{1}{y})^2$. From the bottom of the curve at $y=\frac{1}{4}$ up to the top at $y=1$ we have a volume of $\int_{1/4}^1 \pi\frac{1}{y^2}\,\mathrm{d}y$. We also need to consider the volume of the cylinder beneath the cuved shape. Its height is $\frac{1}{4}$ and its radius is $r=4$ so it has a volume of $\pi 4^2\cdot\frac{1}{4}=4\pi$.
The total volume, then is

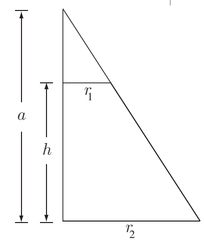
$$V = \int_{1/4}^1 \pi\frac{1}{y^2}\,\mathrm{d}y + 4\pi = 4\pi - \frac{\pi}{y}\Big|_{1/4}^1 = 7\pi.$$

6. Derive the formula for the volume of a frustum where the base radii are r_1 and r_2 and the height is h.



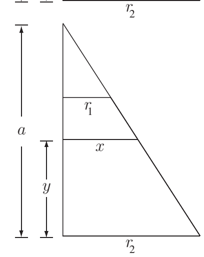
Solution: From the first triangle we have

$$\begin{aligned} \frac{r_2}{a} &= \frac{r_1}{a-h} \\ r_2(a-h) &= a \cdot r_1 \\ a &= \frac{hr_2}{r_2 - r_1} \end{aligned}$$



From the second triangle, we have

$$\begin{aligned} \frac{r_2}{a} &= \frac{x}{a-y} \\ x &= \frac{r_2(a-y)}{a} \end{aligned}$$



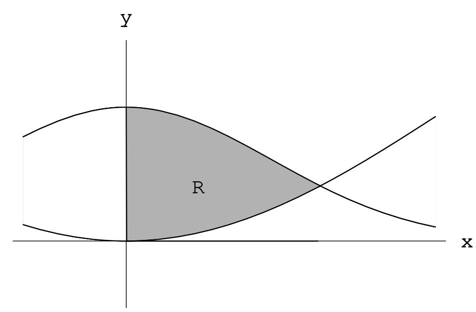
Combining the two results we have: $x = \dfrac{hr_2 - y(r_2 - r_1)}{h}$

The volume of the frustum comes from

$V = \dfrac{\pi}{h^2} \displaystyle\int_0^h [hr_2 - y(r_2 - r_1)]^2 \, \mathrm{d}y$

So $V = \dfrac{1}{3}\pi h \left(r_1^2 + r_1 r_2 + r_2^2\right)$

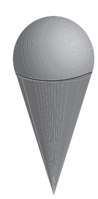
7. The region R bounded by $f(x) = e^{-x^2}$ and $g(x) = 1 - \cos x$ is shown. Write the integral for the volume of the solid generated by revolving R about the x–axis.



Solution: Let b be the solution to $e^{-x^2} = 1 - \cos x$: $b \approx 0.9419$.

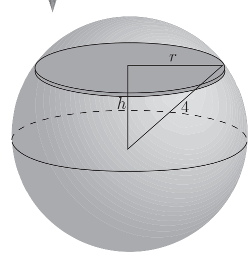
Then $V = \pi \displaystyle\int_0^b (e^{-x^2})^2 - (1 - \cos x)^2 \, \mathrm{d}x$

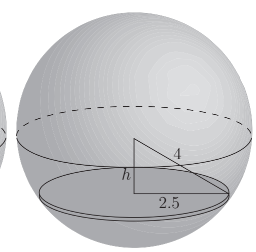
8. An icecream cone has radius 2.5 cm at the top. If a scoop of icecream in the form of a sphere with radius 4cm is placed on top of the cone, what percentage of the icecream is outside the cone?



Solution: The integral looks much like the one for determining the volume of a sphere. (See the first sphere.) The only issue is the bounds of integration and these can be seen in the second figure.

$V = \pi \displaystyle\int_{-\sqrt{9.75}}^{4} 16 - h^2 \, \mathrm{d}h = \pi \left(16h - \frac{1}{3}h^3\right)\Bigg|_{-\sqrt{9.75}}^{4} \approx 259.11$





The percentage is given by dividing by the volume of the sphere $(4/3\pi(4)^3)$. We get about 97%.