

YOU MAY USE A CALCULATOR TO VERIFY SOLUTIONS, BUT NOT TO PROVIDE THEM.

Show all relevant work!

1. Find the values of k or which $y = x^2 + k$ is a solution to the differential equation $2y - xy' = 10$.

Solution:

$$2y - xy' = 10 \quad (1)$$

$$2(x^2 + k) - x(-2x) = 10 \quad (2)$$

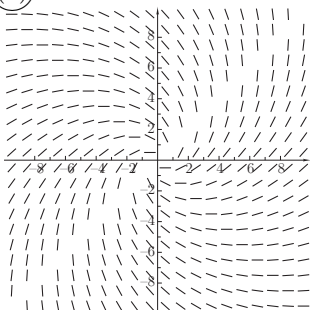
$$2x^2 + 2k - 2x^2 = 10 \quad (3)$$

$$2k = 10 \quad (4)$$

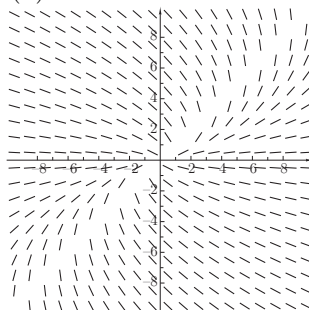
$$k = 5 \quad (5)$$

2. One of the slope fields below has the equation $y' = (x + y)/(x - y)$. Which one?

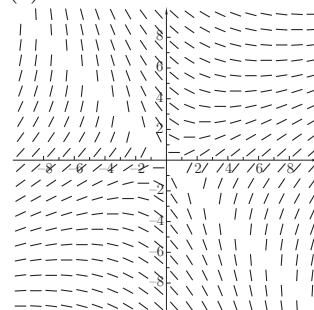
(a)



(b)



(c)

**Solution:** Note the horizontal slopes along the line $y = -x$ (where $x + y = 0$).

3. Water leaks from a vertical cylindrical tank through a small hole in its base at a rate proportional to the square root of the volume of water remaining. If the tank initially contains 200 liters of water and 20 liters leak out during the first day, when will the tank be half empty? How much water will there be after 4 days?

Solution:

$$\frac{dV}{dt} = k\sqrt{V}$$

$$\frac{dV}{\sqrt{V}} = k dt$$

$$\int \frac{dV}{\sqrt{V}} = \int k dt$$

$$2\sqrt{V} = kt + C$$

$$\sqrt{V} = k_1 t + C_1$$

$$V = (k_1 t + C_1)^2$$

$$V(0) = (k_1(0) + C_1)^2$$

$$200 = (C_1)^2$$

$$\sqrt{200} = C_1$$

$$V = (k_1 t + \sqrt{200})^2$$

$$V(1) = (k_1(1) + \sqrt{200})^2$$

$$180 = (k_1(1) + \sqrt{200})^2$$

$$\sqrt{180} = k_1 + \sqrt{200}$$

$$\sqrt{180} - \sqrt{200} = k_1 \rightarrow \approx -0.728$$

$$V = \left[(\sqrt{180} - \sqrt{200})t + \sqrt{200} \right]^2$$

$$\approx (-0.728t + 14.142)^2$$

Half empty:

$$100 \approx (-0.728t + 14.142)^2$$

$$10 \approx (-0.728t + 14.142)$$

$$-4.142 \approx -0.728t$$

$$5.71 \approx t$$

After 4 days:

$$V \approx (-0.728(4) + 14.142)^2$$

$$\approx 126.3$$

126.3 liters remain after 4 days.

So after about 6 days.

4. Any population, P , for which we can ignore immigration, satisfies the differential equation

$$\frac{dP}{dt} = \text{Birth rate} - \text{Death rate.}$$

For organisms which need a partner for reproduction but rely on a chance encounter for meeting a mate, the birth rate is proportional to the square of the population. Thus such an organism satisfies a differential equation of the form

$$\frac{dP}{dt} = aP^2 - bP, \quad a, b > 0$$

The slope field for the equation $\frac{dP}{dt} = aP^2 - bP, \quad a, b > 0$ is shown to the right.

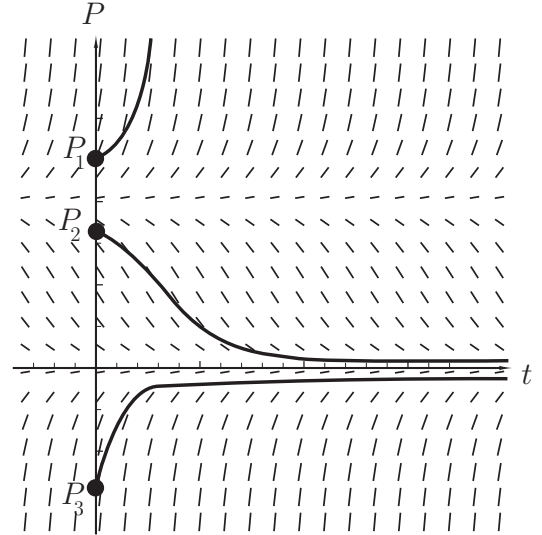
- (a) Determine the equilibrium solutions to this differential equation. State whether they are stable or unstable.

Solution:

$$\begin{aligned} aP^2 - bP &= 0 \\ P(aP - b) &= 0 \\ P &= 0, \quad P = b/a \end{aligned}$$

The slope field approaches $P = 0$ so it is a stable equilibrium. The slope field bends away from $P = b/a$ so it is unstable.

- (b) Sketch representative solution curves for the various initial populations (P_1, P_2 , and P_3) shown.



- (c) Describe the behavior of each population curve you sketched in (b) in terms of the starting population relative to the equilibrium values, its growth rate and its long term outcome. Explain why $P = b/a$ is called the *threshold population*.

Solution:

P_1 is greater than the equilibrium at b/a and grows at an increasing rate without bound. The population grows indefinitely.

P_2 is positive but less than the equilibrium value at $P = b/a$ and decreases towards zero. Eventually this population dies out.

P_3 is a negative population so not an accurate model of reality.

For populations smaller than $P = b/a$ the population cannot sustain itself and ultimately dies off. For populations greater than b/a the population thrives. $P = b/a$ is the minimum population capable of maintaining itself.

5. A mass on a spring oscillates with simple harmonic motion (no friction). If its period is 3 seconds, write the differential equation for this oscillating spring.

Solution: We know that the differential equation for SHO is given by $s'' + \omega^2 s = 0$. The solution to this equation is $s = C_1 \cos(\omega t) + C_2 \sin(\omega t) = \sin(\omega t + \phi)$. We are given the period, $T = 3$ so $\omega = \frac{2\pi}{3}$ and therefore $\omega^2 = \frac{4\pi^2}{9}$. It follows: $s'' + \frac{4\pi^2}{9}s = 0$.

6. A yam is put in a 200°C oven and heats up at a rate proportional to the difference between the temperature, H , of the yam at any time, t , and the temperature of the oven.

(a) If the yam is at 20°C when it is first put into the oven, write and solve a differential equation for the temperature of the yam at any time, t .

(b) Find k using the fact that after 30 minutes the temperature of the yam is 120°C .

Solution:

$$(a) \quad \frac{dH}{dt} = -k(H - 200)$$

$$\frac{dH}{H - 200} = -k dt$$

$$\int \frac{dH}{H - 200} = \int -k dt$$

$$\ln |H - 200| = -kt + C$$

$$|H - 200| = e^{-kt+C}$$

$$H - 200 = Ae^{-kt}, A \in \mathbb{R}$$

$$H = 200 + Ae^{-kt}$$

$$20 = 200 + Ae^0$$

$$-180 = A$$

$$H = 200 - 180e^{-kt}$$

$$(b) \quad H = 200 - 180e^{-kt}$$

$$H(30) = 200 - 180e^{-30k}$$

$$120 = 200 - 180e^{-30k}$$

$$-80 = -180e^{-30k}$$

$$\frac{4}{9} = e^{-30k}$$

$$\ln\left(\frac{4}{9}\right) = -30k$$

$$0.027 \approx k$$

7. Solve the differential equation $x(x+1)\frac{du}{dx} = u^2$ where $u(1) = 1$

Solution:
 $x(x+1)\frac{du}{dx} = u^2$

$$\frac{du}{u^2} = \frac{dx}{x(x+1)}$$

$$\int \frac{du}{u^2} = \int \frac{dx}{x(x+1)}$$

$$\frac{-1}{u} = \int \frac{1}{x} - \frac{1}{(x+1)} dx$$

$$\frac{-1}{u} = \ln|x| - \ln|x+1| + C$$

$$\frac{1}{u} = \ln|x+1| - \ln|x| + C_1$$

$$\frac{1}{u} = \ln\left|\frac{x+1}{x}\right| + C_1$$

$$u = \frac{1}{\ln\left|\frac{x+1}{x}\right| + C_1}$$

$$u(1) = \frac{1}{\ln\left|\frac{2}{1}\right| + C_1}$$

$$1 = \frac{1}{\ln 2 + C_1}$$

$$1 = \ln 2 + C_1$$

$$1 - \ln 2 = C_1$$

$$u = \frac{1}{\ln\left|\frac{x+1}{x}\right| - \ln 2 + 1}$$

$$= \frac{1}{\ln\left|\frac{x+1}{2x}\right| + 1}$$