Modeling Motion

A mathematician is a blind man in a dark room looking for a black cat which isn't there - Charles R. Darwin

Mathematics is inadequate to describe the universe, since mathematics is an abstraction from natural phenomena. Also, mathematics may predict things which don't exist, or are impossible in nature. – Ludovico delle Colombe Criticizing Galileo.

I am too good for philosophy and not good enough for physics. Mathematics is in between.– George Polya

0.1 Horizontal Motion

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0.2 Horizontal Motion

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$$s(t) = v_0 t + s_0 \tag{1}$$

Where v_0 is an object's initial velocity (moving horizontally) and s_0 is the object's starting place measured relative to some position (typically we make $s_0 = 0$).

0.3 Vertical Motion

We've seen that the position of an object launched straight up (or down) is modeled by

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + h_0 \tag{2}$$

where g reflects acceleration due to gravity, v_0 the object's initial velocity, and h_0 the object's initial height relative to some position (usually the ground). The components of this function are reasonably easy to understand when taken one at a time.

0.4 Vertical Motion

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- **Distance related to acceleration from gravity**

  By experiment we can see that the force of attraction between the Earth and objects near its surface accelerates those objects at a constant rate of about 9.81 meters per second per second (abbreviated m/s^2) or equivalently, 32 feet per second per second.
  Why does $-\frac{1}{2}gt^2$ provide the position of an object falling due to gravity?
  Let's consider an object increasing in velocity by 9.8 meters per second every second. The table below gives the velocity of the object at the *end* of each time interval.

t (seconds)	0	1	2	3	4	5	6
$v(t)$ (velocity in m/s)	0	-9.8	-19.6	-29.4	-39.2	-49	-58.8
$v(t)$ (velocity in g)	0	$-g$	$-2g$	$-3g$	$-4g$	$-5g$	$-6g$

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In order to find the distance traveled over each time interval we need the *average* velocity over that time.

For example, between $t = 0$ and $t = 1$ second the object accelerates from 0 to -9.8 meters per second so its average velocity for the first second is $\dfrac{-9.8+0}{2} = -4.9 = \dfrac{1}{2}g$ m/s. Similarly for the next second the average velocity is $\dfrac{-19.6+-9.8}{2} = -14.7 = \dfrac{-3}{2}g$ m/s. The table below continues the pattern of average velocity *over* each interval.

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t (seconds)	$0-1$	$1-2$	$2-3$	$3-4$	$4-5$	$5-6$
$\bar{v}(t)$ (average velocity in m/s)	-4.9	-14.7	-24.5	-34.3	-44.1	-53.9
$\bar{v}(t)$ (avg. velocity in g)	$-\frac{1}{2}g$	$-\frac{3}{2}g$	$-\frac{5}{2}g$	$-\frac{7}{2}g$	$-\frac{9}{2}g$	$-\frac{11}{2}g$

Note that the equation for this table is given by $\bar{v}(t) = -9.8t + 4.9$ or equivalently, $\bar{v}(t) = -gt + \frac{1}{2}g$.

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To find the total distance traveled after t seconds, we take the distance fallen during each interval (1 second times the average velocity) and make a cumulative sum:

t (seconds)	1	2	3	4	5	6
$y(t)$ (position in m)	-4.9	-19.6	-44.1	-78.4	-122.5	-176.4
$y(t)$ (position in terms of g)	$-\frac{1}{2}g$	$-\frac{4}{2}g$	$-\frac{9}{2}g$	$-\frac{16}{2}g$	$-\frac{25}{2}g$	$-\frac{36}{2}g$

While the numbers in the second row may not offer an obvious pattern, the terms in the third row giving the position in terms of g are unmistakably $y = \frac{1}{2}gt^2$.

- **Distance from initial velocity**

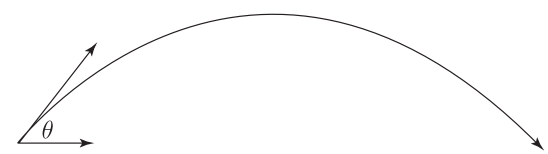
  When a car travels at 60 mph for three hours we compute the net distance travelled by multiplying 60×3. There is no distinction from the $v_0 t$ component of vertical motion. The initial velocity of an object propels it either upwards or downwards and in the absence of gravity (as with horizontal motion), the object's position is simply a product of its constant velocity and the time it has been traveling.

The mathematical representation of vertical motion (2) treats the position of an object over time as though it was composed of discrete position elements. An initial velocity of 40 m/s propelling a ball upwards, for example, so that at each second it is 40m higher, while at the same time gravity pulls it down by -4.9 times the square of the time since it was launched. It gives the impression of a ball simultaneously rising and dropping at every second. While this doesn't accurately describe the action as we see it, it does describe the resulting position at every moment.

0.5 Range: Vertical and Horizontal Motion

While the formula we derived previously (2) accurately describes position over time as long as we can ignore air resistance (so as long as the object is aerodynamic and its initial velocity isn't too great), it limits us to situations where something is launched straight up. In most cases, by design or otherwise, we project an object at some angle other than horizontal or vertical. How do we describe the location in space of an object that is projected into the air at an angle? The most common solution to this question is to break the vector describing the initial velocity down into itsw component horizontal and vertical parts. From there we develop a single formula describing the horizontal distance traveled by the object as a function of its launch angle, θ.

Eventually you will need the identity, $\sin(x + y) = \sin x \cos y + \cos x \sin y$ in the form where $x = y$ so we have:



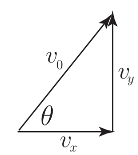
$$\sin 2x = 2 \sin x \cos x \tag{4}$$

0.6 Theory

We've derived formulas to model horizontal (1) and vertical (2) motion, where g reflects acceleration due to gravity, v_x and v_y represent velocity in the horizontal and vertical directions respectively.

$$\begin{aligned} x(t) &= v_x t && (5)\\ y(t) &= -\frac{1}{2}gt^2 + v_y t + y_0 && (6) \end{aligned}$$

We then generalized these into vector component formulas (5 and 6) based on an initial velocity, v_0 and launch angle, θ.



$$\begin{aligned} x(t) &= (v_0 \cos\theta)t && (7)\\ y(t) &= -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + y_0 && (8) \end{aligned}$$

By solving (6) for the time the projectile is in the air and substituting in (5) we derived the range formula relating the horizontal distance, the launch angle, and the initial velocity:

$$x(\theta) = (v_0 \cos\theta)\left(\frac{v_0 \sin\theta + \sqrt{(v_0 \sin\theta)^2 + 2gy_0}}{g}\right) \tag{9}$$

If we assume the starting height is $y_0 = 0$, then we have

$$\begin{aligned} x(\theta) &= (v_0 \cos\theta)\left(\frac{v_0 \sin\theta + v_0 \sin\theta}{g}\right) && (10)\\ &= \frac{v_0^2}{g} \cdot 2\sin\theta\cos\theta && (11) \end{aligned}$$

This is the form described in (4) where $2 \sin\theta \cos\theta = \sin(2\theta)$ and we have

$$x(\theta) = \frac{v_0^2}{g} \sin(2\theta) \tag{12}$$