

The Power Rule

We want to find a general form of the derivative of $f(x) = x^n$. In the case where n is a natural number we can apply the Binomial Theorem to expand the numerator of the difference quotient and derive our desired result. A more general form of the power rule will follow the introduction to the chain rule.

Proof:

Let $f(x) = x^n, n \in \mathbb{N}$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \tag{1}$$

$$= \lim_{h \rightarrow 0} \frac{(x^n + nx^{n-1}h + n(n-1)x^{n-2}h^2 + \dots + \frac{n!}{r!(n-r)!}x^{n-r}h^r + \dots + h^n) - x^n}{h} \tag{2}$$

$$= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + n(n-1)x^{n-2}h^2 + \dots + \frac{n!}{r!(n-r)!}x^{n-r}h^r + \dots + h^n}{h} \tag{3}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{h}(nx^{n-1} + n(n-1)x^{n-2}h + \dots + \frac{n!}{r!(n-r)!}x^{n-r}h^{r-1} + \dots + h^{n-1})}{\cancel{h}} \tag{4}$$

$$= \lim_{h \rightarrow 0} \frac{nx^{n-1} + n(n-1)x^{n-2}h + \dots + \frac{n!}{r!(n-r)!}x^{n-r}h^{r-1} + \dots + h^{n-1}}{1} \tag{5}$$

$$= nx^{n-1} \tag{6}$$