The Power Rule

We want to find a general form of the derivative of $f(x) = x^n$. In the case where n is a natural number we can apply the Binomial Theorem to expand the numerator of the difference quotient and derive our desired result. A more general form of the power rule will follow the introduction to the chain rule.

Proof:
Let $f(x) = x^n, n \in \mathbb{N}$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} && (1)\\
&= \lim_{h\to 0} \frac{\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + \frac{n!}{r!(n-r)!}x^{n-r}h^r + \cdots + h^n\right) - x^n}{h} && (2)\\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + \frac{n!}{r!(n-r)!}x^{n-r}h^r + \cdots + h^n}{h} && (3)\\
&= \lim_{h\to 0} \frac{\not{h}\left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + \frac{n!}{r!(n-r)!}x^{n-r}h^{r-1} + \cdots + h^{n-1}\right)}{\not{h}} && (4)\\
&= \lim_{h\to 0} \frac{nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + \frac{n!}{r!(n-r)!}x^{n-r}h^{r-1} + \cdots + h^{n-1}}{1} && (5)\\
&= nx^{n-1} && (6)
\end{aligned}$$