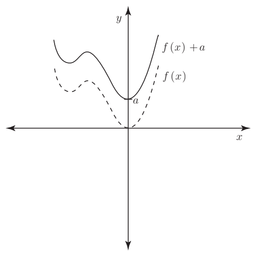
Some Solutions

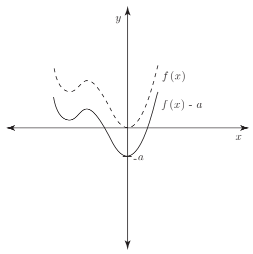
0.1 Shifts and Stretches

Shifts (vertical)

graphic:



$f(x)+a$
$(a>0)$



$f(x)-a$
$(a>0)$

tabular:

x	-2	-1	0	1	2
$f(x)$	4	2	0	2	5

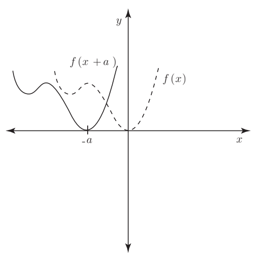
x	-2	-1	0	1	2
$f(x)+a$	6	4	2	4	7

x	-2	-1	0	1	2
$f(x)-a$	2	0	-2	0	3

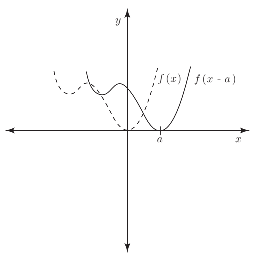
(e.g. $a=2$)

Shifts (horizontal)

graphic:



$f(x+a)$
$(a>0)$



$f(x-a)$
$(a>0)$

tabular:

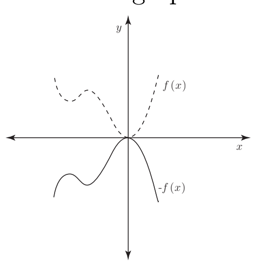
x	-2	-1	0	1	2
$f(x)$	4	2	0	2	5

x	-2	-1	0	1	2
$f(x+a)$	0	2	5		

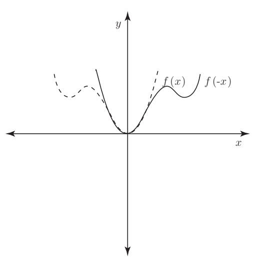
x	-2	-1	0	1	2
$f(x-a)$			4	2	0

(e.g. $a=2$)

Reflections graphic:



$-f(x)$



$f(-x)$

tabular:

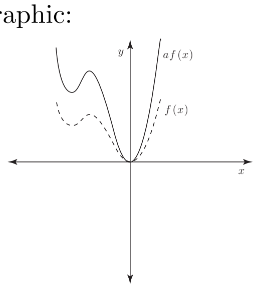
x	-2	-1	0	1	2
$f(x)$	4	2	0	2	5

x	-2	-1	0	1	2
$-f(x)$	-4	-2	0	-2	-5

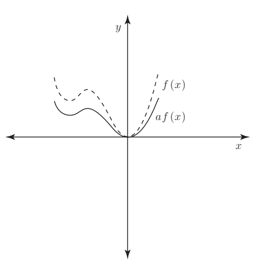
x	-2	-1	0	1	2
$f(-x)$	5	2	0	2	4

Stretches/Compressions (vertical)

graphic:



$af(x)$
$(a>1)$



$af(x)$
$(0<a<1)$

tabular:

x	-2	-1	0	1	2
$f(x)$	4	2	0	2	5

x	-2	-1	0	1	2
$af(x)$	8	4	0	4	10

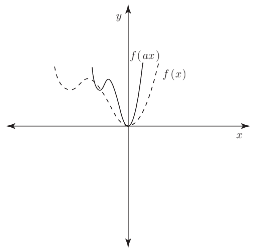
(e.g. $a=2$)

x	-2	-1	0	1	2
$af(x)$	2	1	0	1	2.5

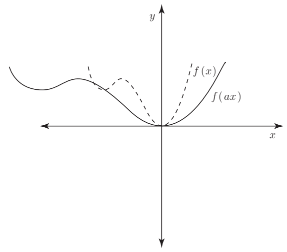
(e.g. $a=\frac{1}{2}$)

Stretches/Compressions (horizontal)

graphic:



$f(ax)$
$(a > 1)$



$f(ax)$
$(0 < a < 1)$

tabular:

x	-2	-1	0	1	2
$f(x)$	4	2	0	2	5

x	-2	-1	0	1	2
$f(ax)$		4	0	5	

(e.g. $a = 2$)

x	-2	-1	0	1	2
$f(ax)$	2		0		2

(e.g. $a = \frac{1}{2}$)

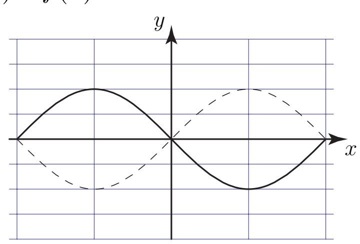
In each of the preceding cases, discuss the effects of the shift or stretch on the average rate of change of the function on the interval $[-2, 2]$.
Note: It's worth some discussion to note which manipulations effect a change in the average rate of change and which don't.

0.2 Odd & Even Functions

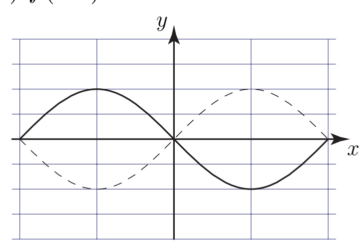
Sketch the following functions.

1. Sketch the indicated reflection over the graph of $g(x)$ in each exercise.

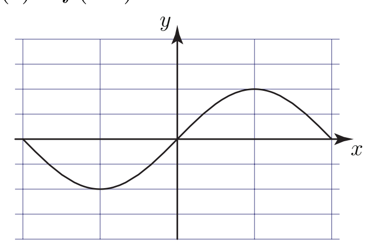
(a) $-f(x)$



(b) $f(-x)$

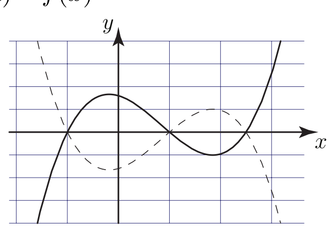


(c) $-f(-x)$

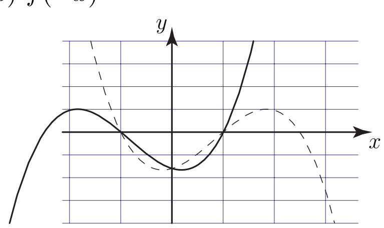


2. Sketch the indicated function over the graph of $f(x)$ in each exercise.

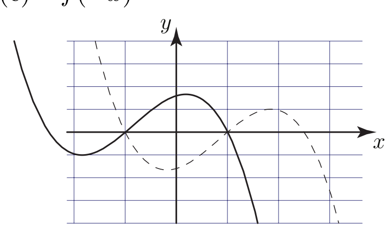
(a) $-f(x)$



(b) $f(-x)$

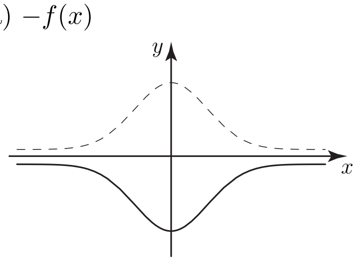


(c) $-f(-x)$

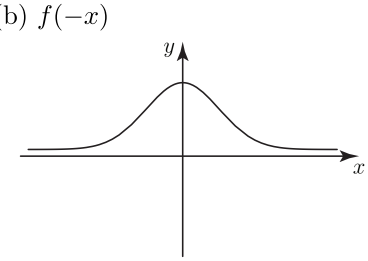


3. Sketch the indicated function over the graph of $f(x)$ in each exercise.

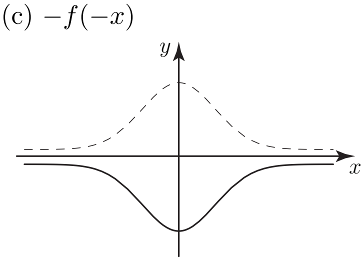
(a) $-f(x)$



(b) $f(-x)$



(c) $-f(-x)$



Questions 0.1

1. Sums. Give either a proof or a counter example for each.

   (a) Is the sum of an odd function and an odd function even? Odd? Neither?

Solution:
Let $h(x) = f(x) + g(x)$ where f and g are both odd.
$h(-x) = f(-x) + g(-x)$
$= -f(x) + -g(x)$ Since both f and g are odd.
$= -(f(x) + g(x))$
$= -h(x)$.
Therefore the sum of two odd functions is odd.

   (b) Is the sum of an odd function and an even function odd? Even? Neither?

Solution:
Neither. For example, if $f(x) = x^2$ and $g(x) = x^3$, then $h(x) = f(x) + g(x) = x^2 + x^3$.
But $h(-1) = 0$ while $h(1) = 2$ so $h(-1) \neq h(1)$ and $h(-1) \neq -h(1)$.

2. Products. Give either a proof or a counter example for each. **See above for similar solutions.**

0.3 Compositions

1. If $h(x) = f(g(x))$, decompose each function into functions f and g where $f(x) \neq x$ and $g(x) \neq x$.

(a) $h(x) = \sin(x^2 + 1)$
$f(x) =$ $\sin x$ $g(x) =$ $x^2 + 1$

(b) $h(x) = (3x - 2)^3$
$f(x) =$ x^3 $g(x) =$ $3x - 2$

(c) $h(x) = \frac{1}{x^2-4}$
$f(x) =$ $\frac{1}{x}$ $g(x) =$ $x^2 - 4$

(d) $h(x) = e^{-x^2}$
$f(x) =$ e^x $g(x) =$ $-x^2$

(e) $h(x) = \frac{1}{\sqrt{3-x}}$
$f(x) =$ $x^{-1/2}$ $g(x) =$ $3 - x$

(f) $h(x) = \frac{1}{x^2} - 4$
$f(x) =$ $x^2 - 4$ $g(x) =$ $\frac{1}{x}$

2. What are the domains of the following functions?
(a) $a(x) = \ln x$ (b) $b(x) = \frac{1}{x^2+1}$ (c) $c(x) = a(b(x))$ (d) $d(x) = b(a(x))$

Solution:
(a) $x \in \mathbb{R}^+$ (b) $x \in \mathbb{R}$ (c) $x \in \mathbb{R}$ (d) $x \in \mathbb{R}^+$

3. (a) On what interval in $[-\pi, \pi]$ is the function $s(x) = x^2$ increasing? On what interval is it decreasing?

Solution: Increasing on $[0, \pi]$, decreasing on $[-\pi, 0]$

(b) On what interval(s) in $[-\pi, \pi]$ is the function $t(x) = \sin x$ increasing? Decreasing?

Solution: Increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, decreasing on $[-\pi, -\frac{\pi}{2}]$ and $[\frac{\pi}{2}, \pi]$.

(c) On what interval(s) in $[-\pi, \pi]$ is the function $w(x) = t(s(x))$ increasing? Decreasing?

Solution: Increasing on $\left[-\pi, -\sqrt{\frac{5\pi}{2}}\right]$, $\left[-\sqrt{\frac{3\pi}{2}}, -\sqrt{\frac{\pi}{2}}\right]$, $[0, \sqrt{\frac{\pi}{2}}]$, and $\left[\sqrt{\frac{3\pi}{2}}, \sqrt{\frac{5\pi}{2}}\right]$.
Decreasing on $\left[-\sqrt{\frac{5\pi}{2}}, -\sqrt{\frac{3\pi}{2}}\right]$, $[-\sqrt{\frac{\pi}{2}}, 0]$, $\left[\sqrt{\frac{\pi}{2}}, \sqrt{\frac{3\pi}{2}}\right]$, and $\left[\sqrt{\frac{5\pi}{2}}, \pi\right]$.

4. The tables of $m(x)$ and $p(x)$ are given below.

x	-6	-4	-2	0	2	4	6
$m(x)$	-1	5	8	7	3	-2	-11

x	-7	-1	5	8	7	3	-2
$p(x)$	-6	-4	-2	0	2	4	6

If $u(x) = p(m(x))$, find . . .

(a) $u(0) =$ 2

(b) $u(2) =$ 4

If $w(x) = m(p(x))$, find . . .

(c) $w(-1) =$ 5

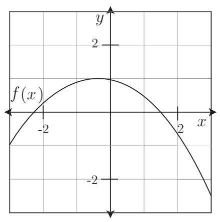
(d) $w(2) =$ Undefined

(e) For what value of x is $w(x) = 7$? At $x = 8$.

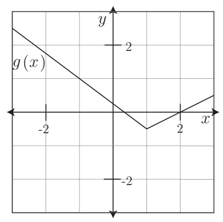
5. Suppose $f(x)$ is an even function and $g(x)$ is an odd function. If $h(x) = f(g(x))$ and $k(x) = g(f(x))$, comment on whether they are even, odd, or neither.

Solution: Both are even functions. $h(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = h(x)$, and $k(-x) = g(f(-x)) = g(f(x) = k(x)$.

6. Use the functions, $f(x)$ and $g(x)$ graphed below to help you sketch $h(x) = f(g(x))$.



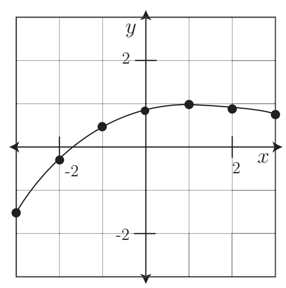
$f(x)$



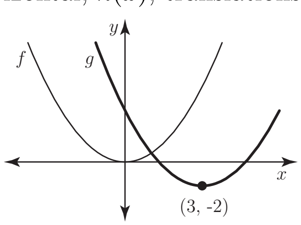
$g(x)$

Solution: Generating the table below helps to produce the graph.

x	-3	-2	-1	0	1	2	3
$g(x)$	2.5	1.8	1	0.2	-0.4	0	0.5
$f(g(x))$	-1.5	-0.3	0.5	0.9	1	0.95	0.8



7. $g(x)$ is a translation of the parabola $f(x)$ as shown below. Write functions for vertical, $v(x)$, and horizontal, $h(x)$, translations and express $g(x)$ as a composition in terms of f, h, and v.



$h(x) =$ $x - 3$

$v(x) =$ $x - 2$

$g(x) =$ $v(f(h(x)))$

0.4 Inverses

1. Are the functions describing these situations invertible?

(a) The temperature of an oven as a function of the time since it was turned on.

Solution: While the oven is heating up, yes. Once it levels off to a constant temperature, no.

(b) The number of people on a bus as a function of the time of day.

Solution: Generally, no. Since the population rises and falls with time, there are probably many times when the population of the bus will be the same.

2. For $g(x) = 10^x - x$, find $g^{-1}(997)$.

Solution: Since $g(3) = 997$, it follows that $g^{-1}(997) = 3$.

3. For each function below, identify a proper domain and give the inverse function.

a) $f(x) = e^{1-3x}$ b) $g(x) = x^2 - 1$ c) $h(x) = \dfrac{x}{x+1}$

Solution:

(a) $f(x)$:
Domain: $x \in \mathbb{R}$
Range: $y \in \mathbb{R}^+$

$f^{-1}(x) = \dfrac{1 - \ln x}{3}$
Domain: $x \in \mathbb{R}^+$
Range: $y \in \mathbb{R}$

To find f^{-1}:
$y = e^{1-3x}$
$x = e^{1-3y}$
$\ln x = 1 - 3y$
$\ln x - 1 = -3y$
$\frac{1-\ln x}{3} = y$

(b) $g(x)$:
Domain: $x \in \mathbb{R}$
Range: $y \geq -1$

$g^{-1}(x) = \sqrt{x+1}$
Domain: $x \geq -1$
Range: $y \geq 1$

To find g^{-1}:
$y = x^2 - 1$
$x = y^2 - 1$
$x + 1 = y^2$
$\sqrt{x+1} = y$

(c) $h(x)$:
Domain: $x \neq -1$
Range: $y \geq -1$

$h^{-1}(x) = \dfrac{x}{1-x}$
Domain: $y \geq -1$
Range: $x \neq -1$

To find h^{-1}:
$y = \frac{x}{x+1}$
$x = \frac{y}{y+1}$
$x(y+1) = y$
$xy + x = y$
$x = y - xy$
$x = y(1-x)$
$\frac{x}{1-x} = y$