Logarithms

0.1 Opener

Solve: (a) $2^x = 8$ (b) $8^x = 32$

$$\begin{aligned} 2^x &= 2^3 \\ \longrightarrow x &= 3 \end{aligned}$$

$$\begin{aligned} (2^3)^x &= 2^5 \\ 2^{3x} &= 2^5 \\ \longrightarrow 3x &= 5 \\ x &= \frac{5}{3} \end{aligned}$$

Moral: If the bases are the same, we can focus on the exponents.

0.2 Other Bases

What if we had to solve $17^x = 40$? Unlike the previous examples, the bases aren't obvious. If we *could* write both numbers as powers of the same base, we'd be in good shape.

0.2.1 History

Although their purposes were different from ours, the Arabs of the thirteenth century and Europeans of the seventeenth century, gave us the tools to do this. John Napier, while not the first to come up with the idea of writing numbers as powers, was the first to publish (around 1614 - he died in 1617). Unfortunately, he chose a really weird base - 0.9999999, and this made for some strange results. His friend, Henry Briggs, from Oxford, published a number of tables shortly after Napier's and these gave all the numbers as powers of various integers less than 1000 (accurate to 8 and later, 14, decimal places)! This is much like the trig tables produced from the same period. Today we use only a few of these bases because calculators and computers make it easy to change bases. In particular, the base 10 power table is stored in our calculators, much like the trig tables.

Because these tables consist entirely of powers of the same base, it's easier just to give the exponents with the base implied. Napier gave these numbers the name **logarithm** from the Greek, *logos*, meaning proportion, and *arismos*, or number. What powers have to do with proportions will be clearer later.

To see a piece of one of these tables, find the log key on your calculator and type in log(17). What is this number? It should be the exponent we raise 10 to in order to get 17. Let's test it.

Type in [2nd] [log] to get 10^ and press [2nd] [(-)] to put in your last answer.

We should have gotten 17 back and this tells us that $\log 17$ is the exponent I raise 10 to in order to get 17. Just like $\sin 30°$ is the number get when I divide the opposite side by the hypotenuse in a $30°$ right triangle.

This gives us all we need to solve $17^x = 40$:

In Decimal Approximation

$$\begin{aligned} (10^{1.230})^x &\approx 10^{1.602} \\ 10^{1.230x} &\approx 10^{1.602} \\ \longrightarrow 1.230x &\approx 1.602 \\ x &\approx 1.302 \end{aligned}$$

In Symbols

$$\begin{aligned} (10^{\log 17})^x &= 10^{\log 40} \\ 10^{(\log 17)x} &= 10^{\log 40} \\ \longrightarrow (\log 17)x &= \log 40 \\ x &= \frac{\log 40}{\log 17} \end{aligned}$$

More Problems:

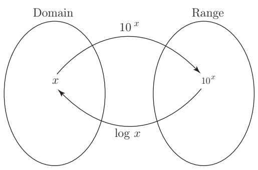
(a) SMCU offers a 6% CD with a minimum investment of \$2,000 for at least 8 months. What would be the interest earned at these minimum amounts (compounded monthly)?

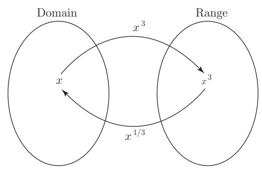
(b) After how long will your \$2,000 double?

(c) In their offer, SMCU claims that their 6% is APR and the nominal percentage is 5.84%. If you compound this percentage monthly do you get an equivalent APR of 6%?

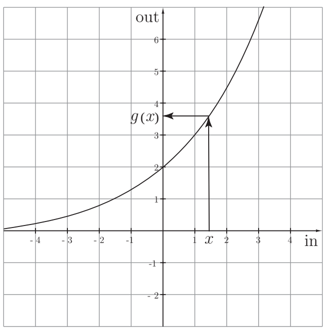
0.3 Inverses

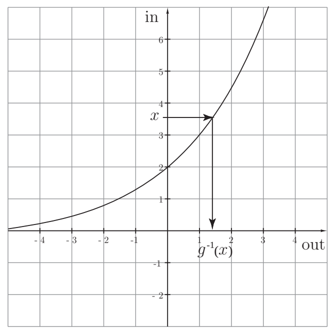
Note symbolically that this reinforces something we already sense - that the log function and the exponential function are inverses (much like $y = x^3$ and $y = \sqrt[3]{x}$):



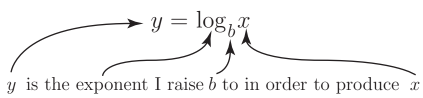


Equivalently,





(Note since $y = 10^x$ is 1:1, it is invertible and the domain and range are switched exactly). Switching the in and out axes will produce the graph of the inverse function in the customary orientation, but this graph gives us a good sense of the log function's behavior.
The general notation for logarithms is written: $y = \log_b x$ and read, y is the exponent I raise b to in order to produce x.



Note the equivalent form in exponential notation can be obtained by writing:

$$\log_b x = y \longrightarrow b^y = x$$

0.4 Applications

From our previous experiences we've seen that populations and bank accounts are typically modeled with exponential functions of the form $P = P_0(1 + \frac{r}{n})^{nt}$. Suppose you have \$1000 invested at 6% compounded quarterly. How long will you have to wait until the principal doubles (or reaches some other amount)? Equivalently, what interest rate would you need to double your money in 5 years?
Note that we can try the table function of the calculator to apply the idea of an inverse function by reading the table backwards. However, this only approximates the table and we want something a little more exact so we apply the same methods as we used to solve $17^x = 40$.

0.4.1 Exercises

1. If you invest \$1200 at 5% compounded quarterly, how long will it be before your investment is worth \$5000?

2. If a population grows from 2,000 in 1990 to 7,000 in 2005, write an equation for the population as a function of time since 1990 and determine both the continuous and annual growth rates for this population.

3. The population of bacteria in a dish doubles in 5 hours. Assuming their growth is exponential, how long will it take the population to triple?

4. In the early 1960's, radioactive strontium-90 was released during atmospheric testing of nuclear weapons and got into the bones of people alive at the time. If the half-life of strontium-90 is 29 years, what fraction of the strontium-90 absorbed in 1960 remained in people's bones in 2000?

0.5 Log Properties

The following relationships are equivalences between logarithmic and exponential notation.

$$10^x \cdot 10^y = 10^{x+y} \longleftrightarrow \log(AB) = \log A + \log B \tag{1}$$

$$\frac{10^x}{10^y} = 10^{x-y} \longleftrightarrow \log(A/B) = \log A - \log B \tag{2}$$

$$(10^x)^n = 10^{nx} \longleftrightarrow \log(A^n) = n \log A \tag{3}$$

We will discuss the first relationship explicitly and then you will be asked to investigate the other two.