

Honors Project 2

I. Do problem #2 on page 114.

II. Complete the following project:

Recall the Power Rule, $f(x) = x^n, n \in \mathbb{N}, f'(x) = nx^{n-1}$. The restriction that $n \in \mathbb{N}$ is a consequence of the Binomial Theorem used in the proof. In this project you will explore integer and rational exponents so that we can extend the power rule to $n \in \mathbb{Q}$ (the rationals).

(i) The first step in this process is to show that the Power Rule applies to $n \in \mathbb{Z}$ (the integers). This requires you to show that for $f(x) = x^n, f'(x) = nx^{n-1}$ for negatives of the natural numbers. The proof is similar to the proof for the naturals using the Binomial Theorem.

(ii) The next step is to show that $f(x) = x^{p/q}, p, q \in \mathbb{Z}, f'(x) = \frac{p}{q}x^{\frac{p}{q}-1}$. In order to do this, you should first study some examples.

(a) Consider the function $f(x) = x^{3/2}$.

You will find it advantageous to work with the simpler elements of this function, namely $y = x^3$ and $y = x^{1/2}$. Find the derivatives of these functions separately from the definition of the derivative function. Then combine your results and possibly some properties of limits to derive the desired result.

(b) Repeat (a) for the function $f(x) = x^{2/3}$.

Again, work with the simpler elements of this function, namely $y = x^2$ and $y = x^{1/3}$. See the commentary below for assistance with the derivative of $y = x^{1/3}$.

Conclude with a proof that for $f(x) = x^{p/q}, p, q \in \mathbb{Z}(q \neq 0), f'(x) = \frac{p}{q}x^{\frac{p}{q}-1}$. It may be of some use to recall from long division that $\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + \dots + x^{n-r}y^r + \dots + xy^{n-2} + y^{n-1}$ so $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \dots + x^{n-r}y^r + \dots + xy^{n-2} + y^{n-1})$.

Completed Project:

When you have done the work necessary to complete the project, you need to prepare it in written form. The paper you turn in should have a mix of equations formulas and prose to support your conclusions. Use complete sentences, good grammar and correct punctuation. The prose should be written in order to convey to the reader an explanation of what you have done. It should be written in such a way that it can be read and understood by anyone who knows the material in this course. You will be graded on your written presentation as well as the mathematical content.

Adapted from Cohen, Gaughan, et al. *Student Research Projects in Calculus*;
MAA: Washington, D.C., 1991