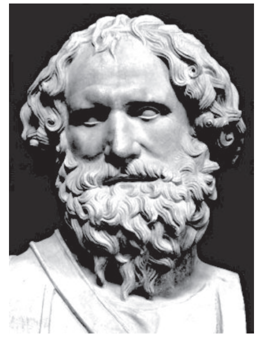
Math Guys - the Early Years

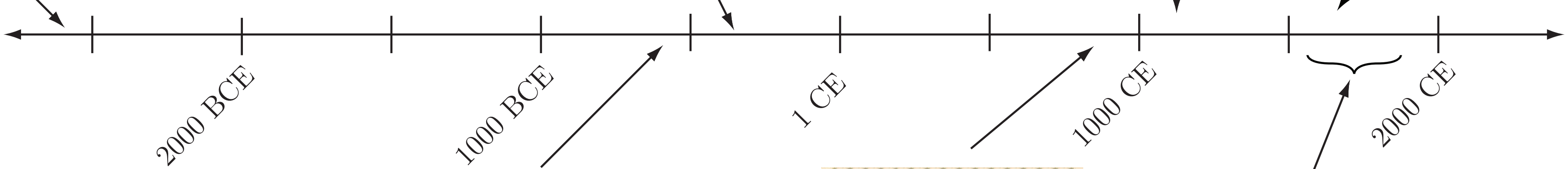
Pyramid of Giza (2551 BCE)

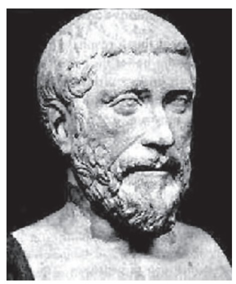


Archimedes 287 BCE

Bhaskara 1114

Newton's
Anni Mirabiles
1664 - 1666

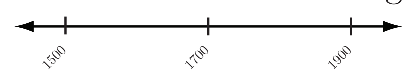




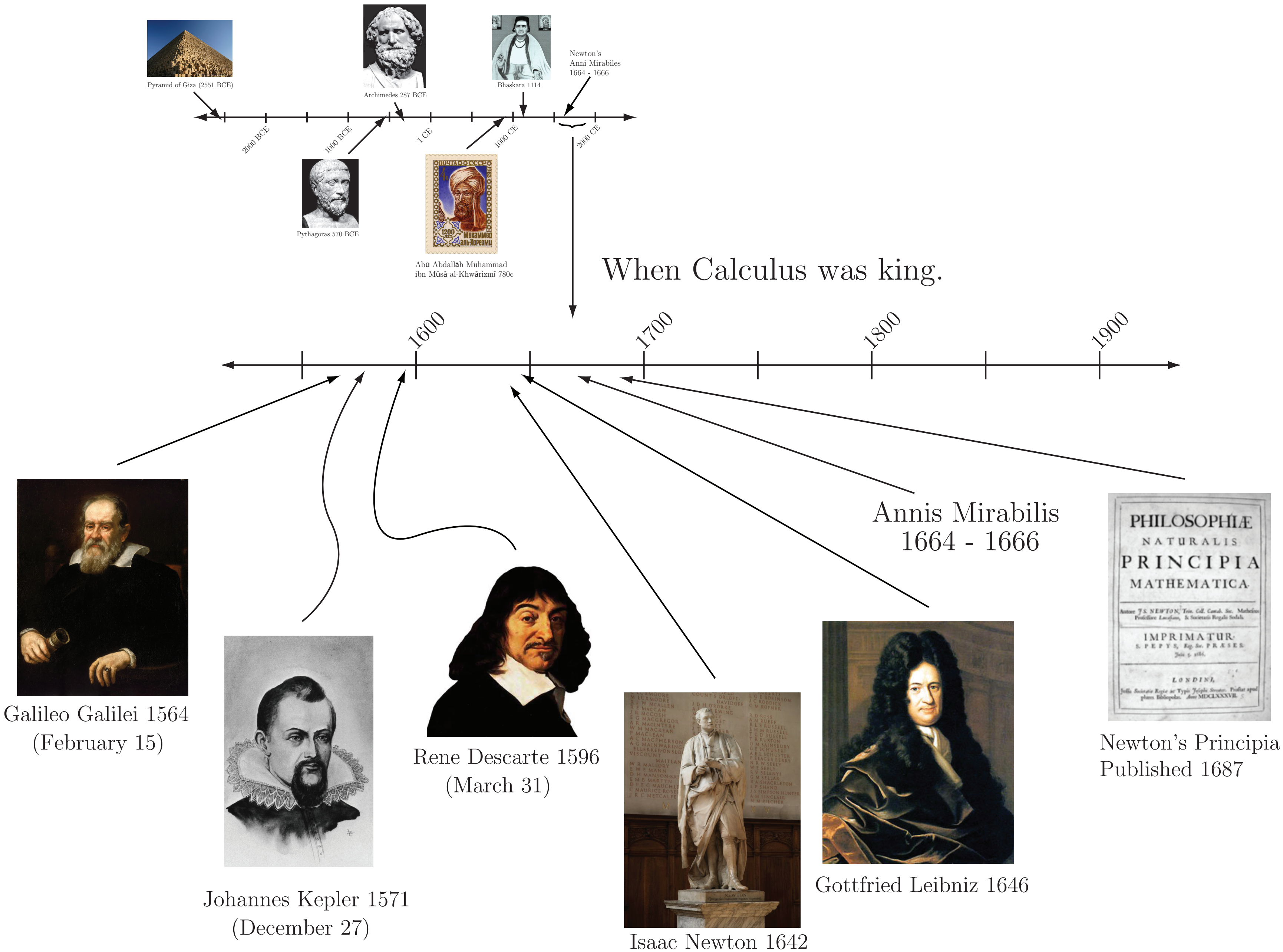
Pythagoras 570 BCE

Abū Abdallāh Muhammad
ibn Mūsā al-Khwārizmī 780c

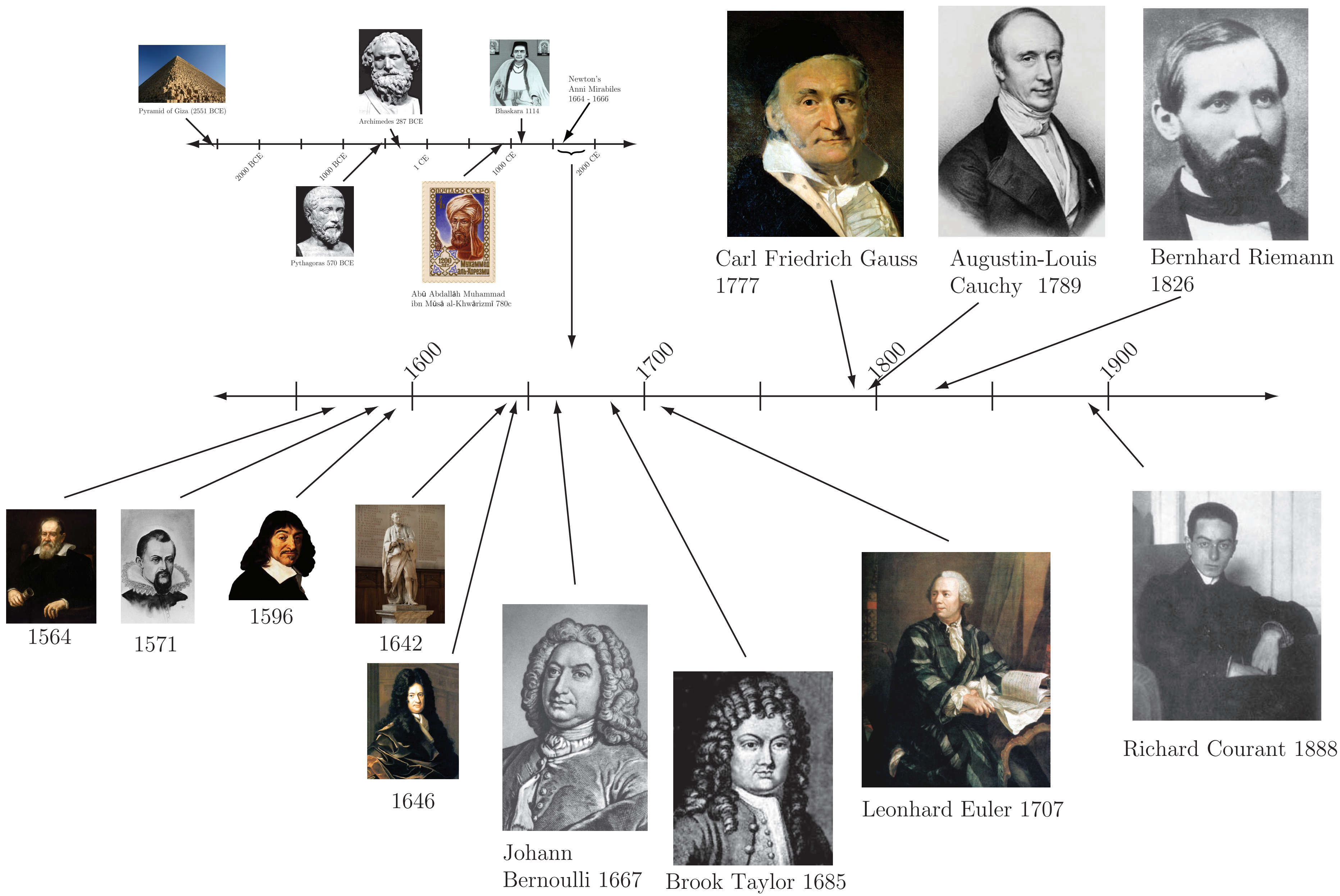
When Calculus was king



The Shoulders of Giants

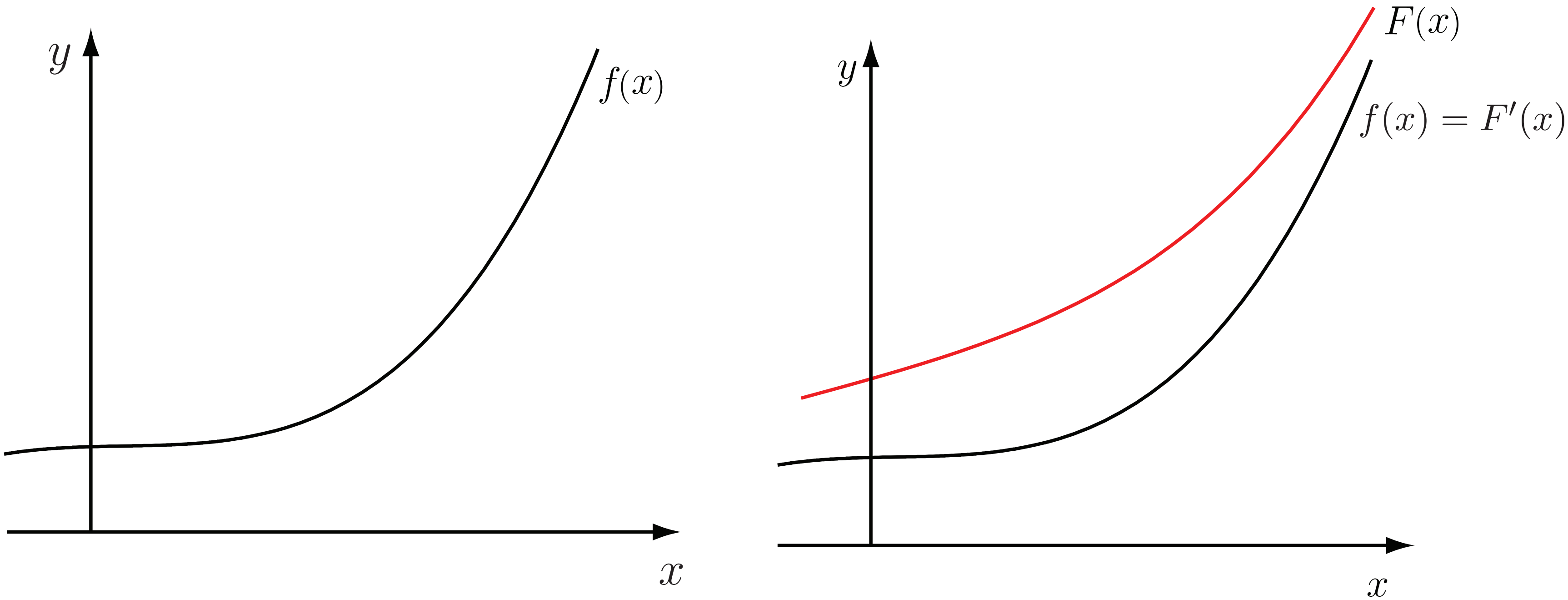


The Calculus Party



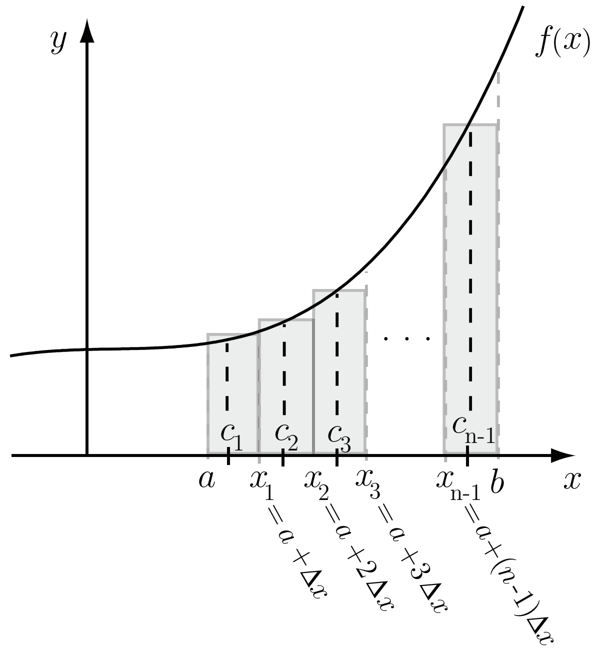
The Fundamental Theorem of Calculus

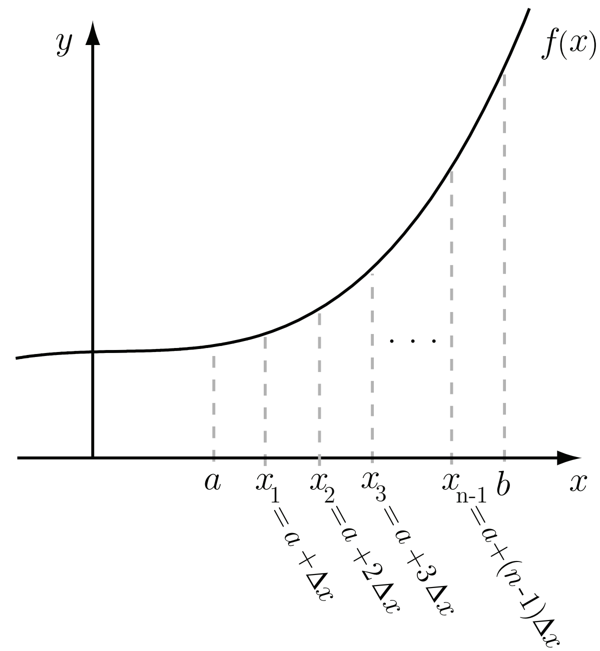
Let $F'(x) = f(x)$, so f is the derivative of F or F is an antiderivative of f.

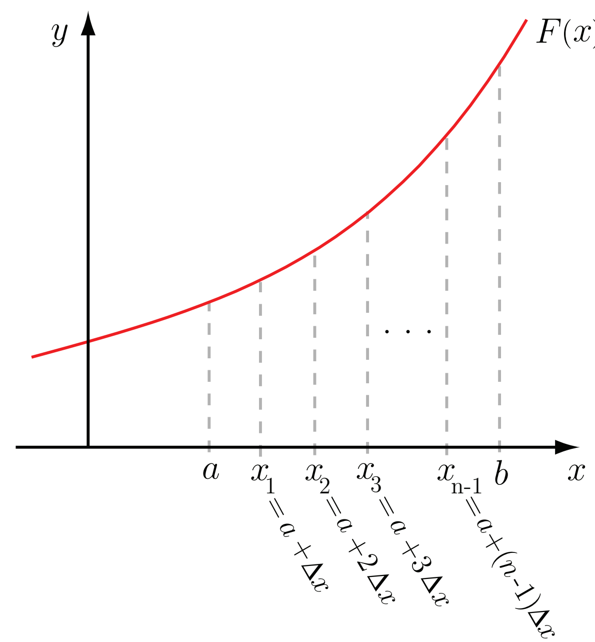


Consider $\int_a^b f(x)\,\mathrm{d}x$. We want to approximate the integral by taking rectangular strips but rather than using the left or right hand sides of the rectangles, we will be more arbitrary.

First note how the intervals of f fit the graph of F.

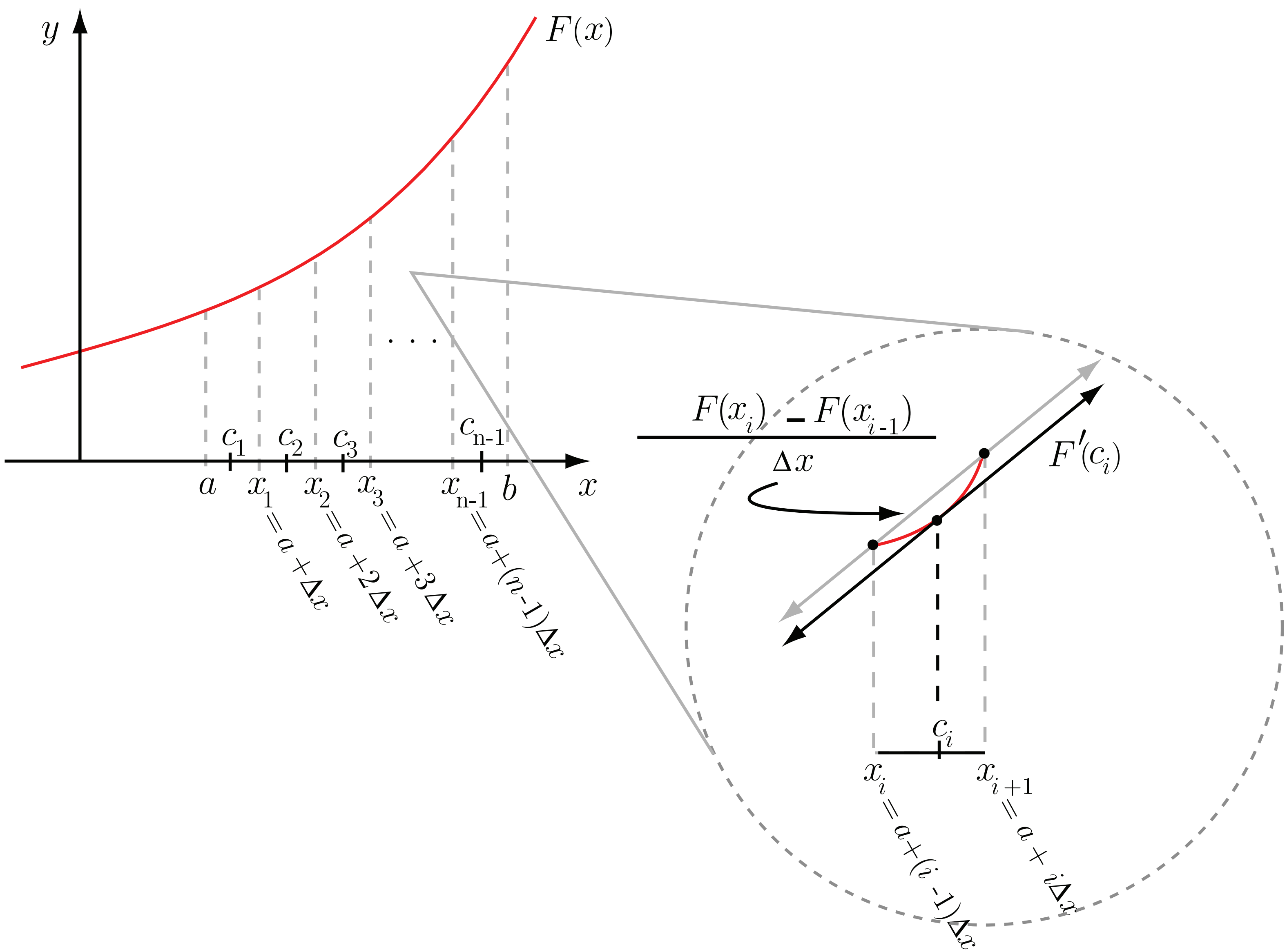






Then, since we assume that $F(x)$ is differentiable it follows that the MVT applies and on every subinterval $[x_{i-1}, x_i]$ of $[a, b]$ there is a point c_i such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{\Delta x}$$



So we generate the list:

$$F'(c_1) = \frac{F(x_1) - F(a)}{\Delta x}$$

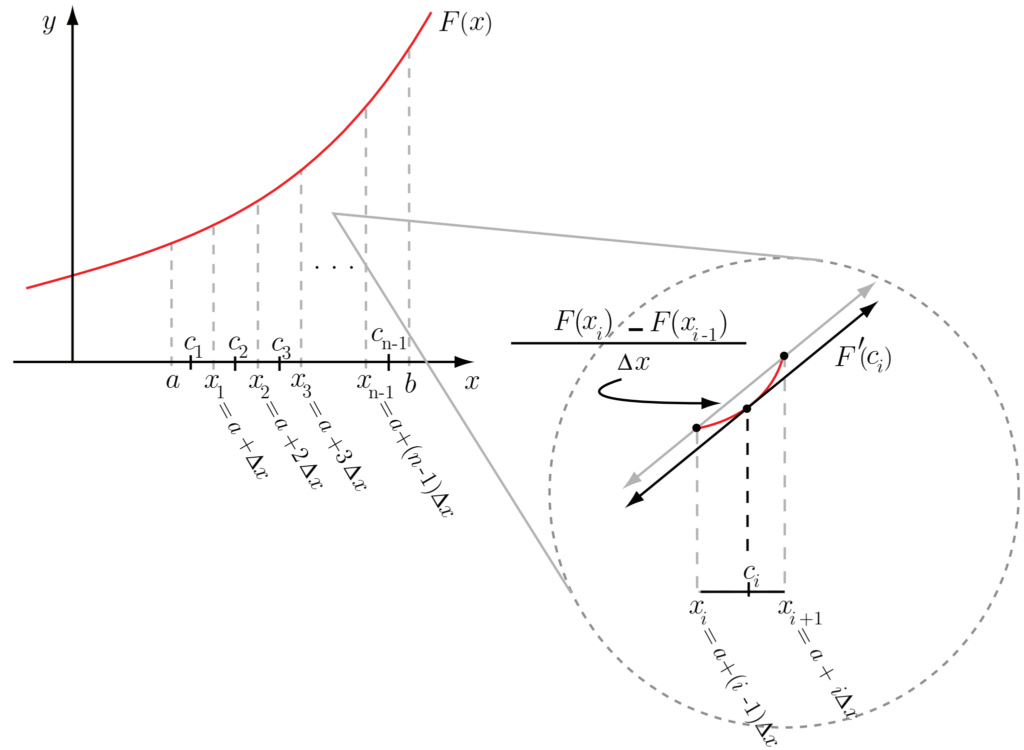
$$F'(c_2) = \frac{F(x_2) - F(x_1)}{\Delta x}$$

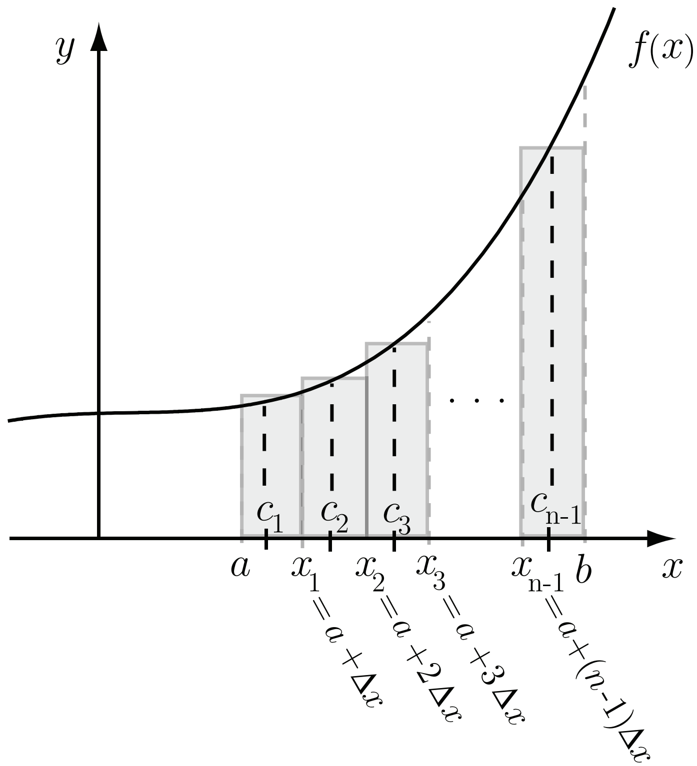
$$F'(c_3) = \frac{F(x_3) - F(x_2)}{\Delta x}$$

$$\vdots$$

$$F'(c_{n-1}) = \frac{F(x_{n-1}) - F(x_{n-2})}{\Delta x}$$

$$F'(c_n) = \frac{F(b) - F(x_{n-1})}{\Delta x}$$



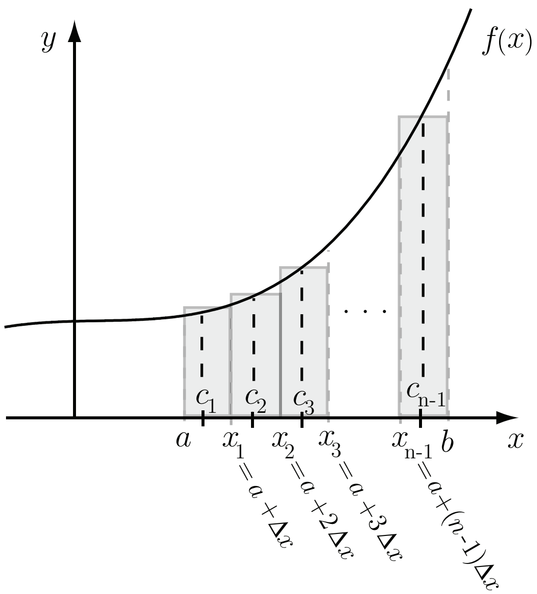


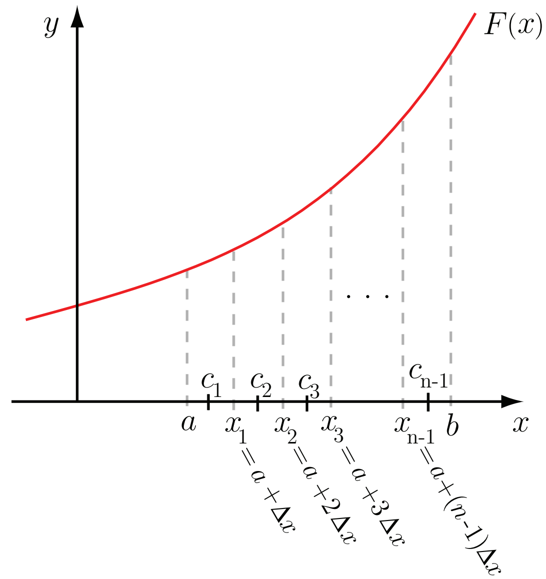
Now to approximate the integral, $\int_a^b f(x)\,\mathrm{d}x$ we can use a Riemann sum of the form

$$R_n = f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_{n-1})\Delta x + f(c_n)\Delta x$$

But since $F'(x) = f(x)$ it follows that

$$R_n = F'(c_1)\Delta x + F'(c_2)\Delta x + \cdots + F'(c_{n-1})\Delta x + F'(c_n)\Delta x$$





$$
\begin{aligned}
F'(c_1) &= \frac{F(x_1) - F(a)}{\Delta x} \\
F'(c_2) &= \frac{F(x_2) - F(x_1)}{\Delta x} \\
F'(c_3) &= \frac{F(x_3) - F(x_2)}{\Delta x} \\
&\vdots \\
F'(c_{n-1}) &= \frac{F(x_{n-1}) - F(x_{n-2})}{\Delta x} \\
F'(c_n) &= \frac{F(b) - F(x_{n-1})}{\Delta x}
\end{aligned}
$$

We have $\int_a^b f(x)\,\mathrm{d}x \approx R_n = F'(c_1)\Delta x + F'(c_2)\Delta x + \cdots + F'(c_{n-1})\Delta x + F'(c_n)\Delta x$

From the MVT list we can substitute $\frac{F(x_i)-F(x_{i-1})}{\Delta x}$ for each $F'(c_i)$ and we have

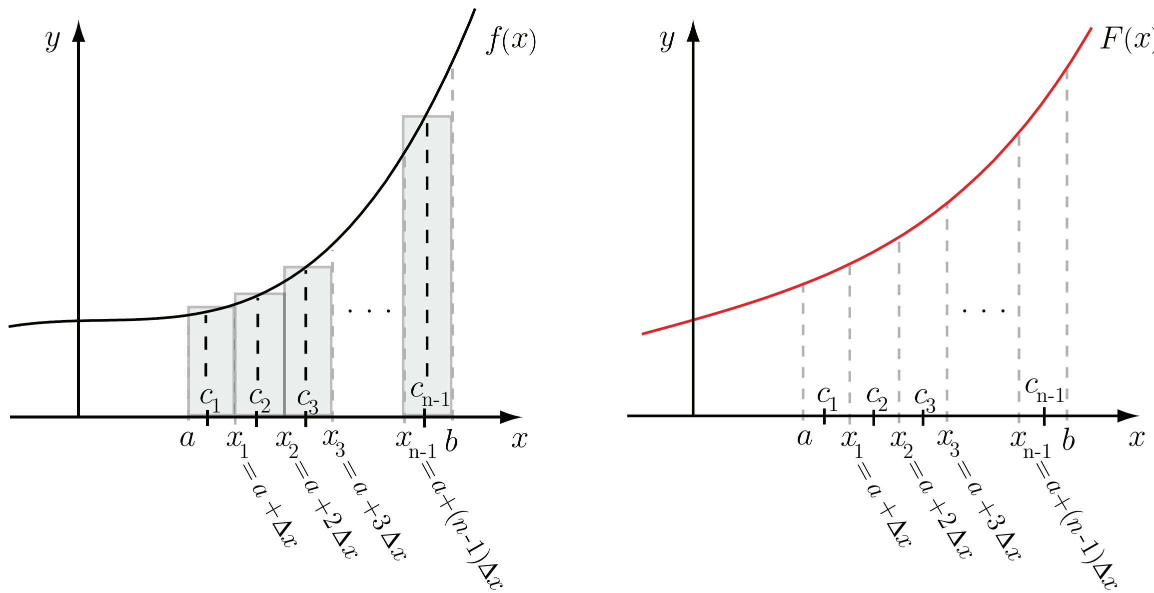
$$
R_n = \frac{F(x_1) - F(a)}{\Delta x} \cdot \Delta x + \frac{F(x_2) - F(x_1)}{\Delta x} \cdot \Delta x + \cdots
$$

$$
\cdots + \frac{F(x_{n-1}) - F(x_{n-2})}{\Delta x} \cdot \Delta x + \frac{F(b) - F(x_{n-1})}{\Delta x} \cdot \Delta x
$$

Cancelling Δx gives

$$R_n = F(x_1) - F(a) + F(x_2) - F(x_1) + F(x_3) - F(x_2) + \cdots$$

$$\cdots + F(x_{n-1}) - F(x_{n-2}) + F(b) - F(x_{n-1})$$



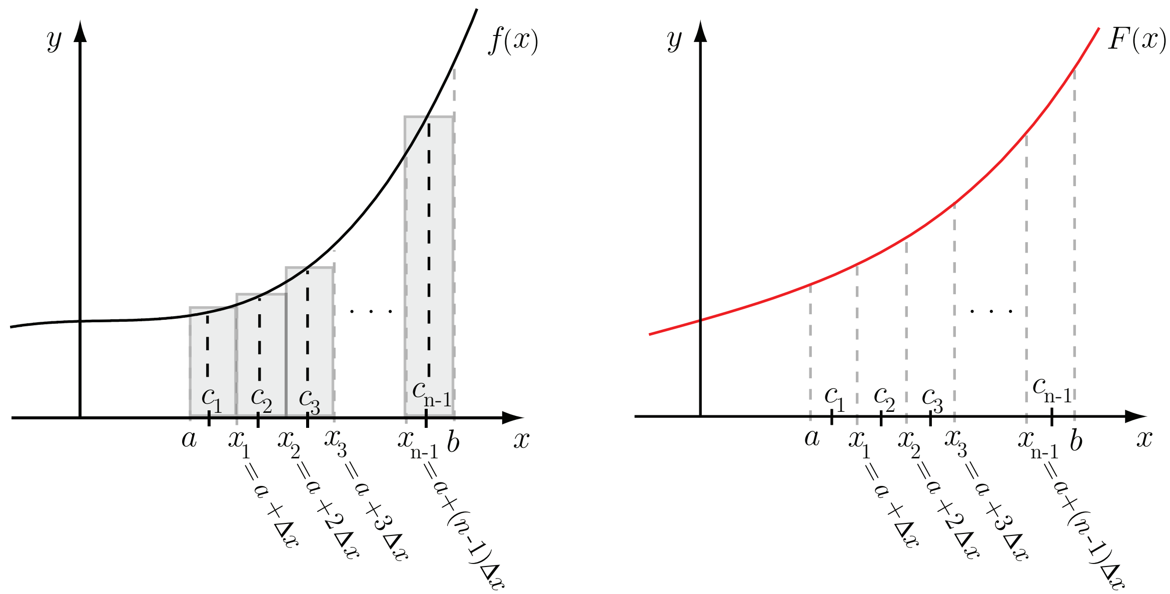
$$R_n = F(x_1) - F(a) + F(x_2) - F(x_1) + F(x_3) - F(x_2) + \cdots + F(x_{n-1}) - F(x_{n-2}) + F(b) - F(x_{n-1})$$

Now note that all of the middle terms drop out: $F(x_1)$ is matched by $-F(x_1)$, $F(x_2)$ is matched by $-F(x_2)$ and so on. Once the middle terms have eliminated each other we are left with

$$R_n = F(b) - F(a)$$

It's hard to appreciate what this gives you until you consider the implications of the right hand side.

R_n is the approximation for the integral, $\int_a^b f(x)\,\mathrm{d}x$ and as we increase the number of intervals (as $n \to \infty$), R_n approaches the actual value of $\int_a^b f(x)\,\mathrm{d}x$.



But $R_n = F(b) - F(a)$ and the right hand side of this equation is independent of n so we have

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} R_n = F(b) - F(a)$$

or more succinctly,

$$\boxed{\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)}$$

What does this mean?

For example, if $f(x) = F'(x) = x^2$, then since $F(x) = \frac{1}{3}x^3$ is an antiderivative of f, it follows that

$$\begin{aligned} \int_0^3 x^2 \, dx &= \left. \frac{1}{3}x^3 \right|_0^3 \\ &= \frac{1}{3}(3)^3 - \frac{1}{3}(0)^3 \\ &= 9 \end{aligned}$$

So the area bounded by $f(x) = x^2$ between $x = 0$ and $x = 3$ (and the x-axis) is 9 square units.