Show

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\ln x &= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} && (1)\\
&= \lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h} && (2)\\
&= \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) && (3)\\
&= \lim_{h\to 0}\ln\left(\frac{x+h}{x}\right)^{1/h} && (4)\\
&= \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{1/h} && (5)
\end{aligned}$$

Now consider the form $\frac{h}{x}$. We recognize this limit is reminiscent of the limit defining e^x but the term isn't quite right. We proceed in two stages. First by getting the exponent to look more like the form $\frac{h}{x}$:

$$\begin{aligned}
\lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{1/h} &= \frac{x}{x}\lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{1/h} && (6)\\
&= \frac{1}{x}\lim_{h\to 0}x\ln\left(1+\frac{h}{x}\right)^{1/h} \qquad \text{(since } x \text{ not a function of } h\text{)} && (7)\\
&= \frac{1}{x}\lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{x/h} && (8)
\end{aligned}$$

The second stage is to consider the term $\frac{x}{h}$.
As $h\to 0$ we have $\frac{x}{h}\to\infty$ so if $H=\frac{x}{h}$ then as $h\to 0$, $H\to\infty$ so we have

$$\begin{aligned}
\frac{1}{x}\lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{x/h} &= \frac{1}{x}\lim_{H\to\infty}\ln\left(1+\frac{1}{H}\right)^{H} && (9)\\
&= \frac{1}{x}\ln\lim_{H\to\infty}\left(1+\frac{1}{H}\right)^{H} && (10)\\
&= \frac{1}{x}\ln e && (11)\\
&= \frac{1}{x} && (12)
\end{aligned}$$