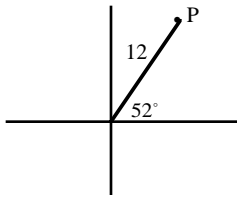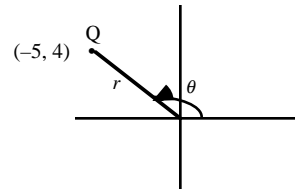# Math 130

Notes: Vectors

Open: a) Find the coordinates of P:

Answer: (7.4, 9.5)

b) If the coordinates of Q are (–5, 4), find r and θ.


Answer: $r = \sqrt{41} \approx 6.4, \quad \theta = 180 - \arctan\left(\frac{4}{5}\right) \approx 141°$

I. Vectors

Vectors give us a different way of thinking about force, velocity and acceleration (among other things). Vectors are used to describe anything with both magnitude (size . . .) and direction. Force is a vector since it involves both magnitude (how hard you push) and direction (pushing to the side versus pushing down). In the case of velocity, its magnitude is commonly called speed but to distinguish it, we note that if North is positive and South is negative, then a 70 mph trip to SF and a 70 mph trip to SJ both have speed 70 but the trip to SF has velocity 70 while the one to SJ (the opposite direction)has velocity –70. Gravity (between Earth and things on or near Earth) is the most common acceleration vector. It has magnitude 32 ft/sec per second (or 9.81 m/sec per second) and its sign is negative to indicate the attraction downward towards the center of the planet.

The geometric representation of a vector is an arrow whose tail is proportional to its magnitude and whose direction is measured relative to the x-axis. We label it $\vec{v}$ or $\mathbf{v}$ (but bold is for books).
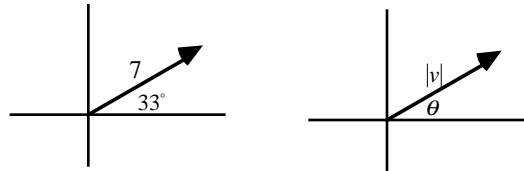
In general we represent the magnitude of a vector, v, as $|v|$ and the direction with a typical angle variable like θ.


Generally we will be interested in breaking a vector up into its horizontal and vertical components. Think of an Etcha-Sketch and how you would turn both knobs to make a diagonal line.

Example: If a football is kicked from the ground at an angle of 28° with initial velocity 30 ft/sec, then we break down the velocity vector into its vertical and horizontal components much like we wrote coordinates for P in the opener.

$\vec{v}_x = 30\cos(28°)$

$\vec{v}_y = 30\sin(28°)$

Note that the sum of the two components is NOT 30!
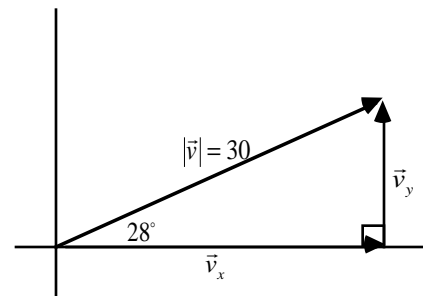
Force works the same way. If you've ever pushed a car on level ground, except for getting it started and overcoming the friction between the parts, once it starts rolling it's pretty easy to keep it going. Put the car on a hill and it's much harder because you have to overcome the component of gravity that is directed down the hill.

Example: Imagine sliding a 100 lb. block up a 25° ramp (see Figure 1). Recall that force in standard units is measured in pounds. In order to move the block you must overcome the force of gravity that pulls the block down the ramp (parallel to the plane of the ramp), this is labeled $F_\parallel$ in the diagram. Don't confuse this with the force the block exerts downward on the ramp, $F_\perp$, (this would be the force you have to overcome in order to lift the block perpendicular to the surface).

Figure 1   Figure 2   Figure 3
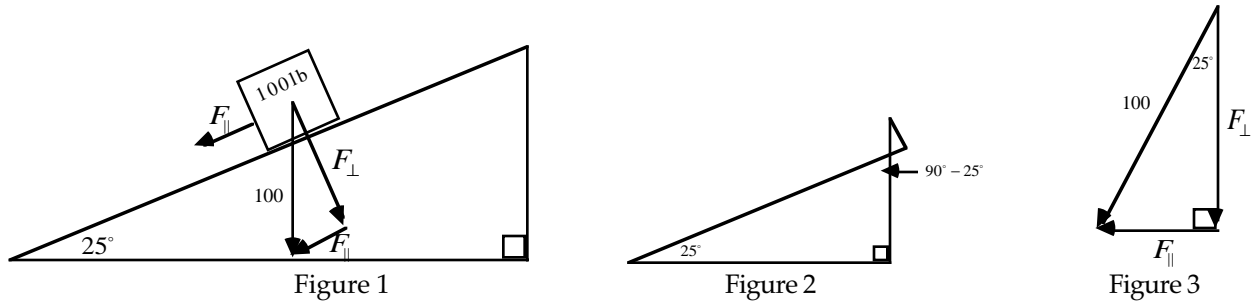
From Figure 1 we can isolate the similar triangles shown in Figure 2 and it follows that the angle from the center of the block is 25°. From this we can break the weight of the box down into its component vectors (Figure 3) and it follows that the force of gravity that pulls the block down the ramp, $F_\parallel$, is given by $F_\parallel = 100\sin 25° \approx 42.26\,$lb.'s. So in order to push the block up the ramp you must exert at least 42.26 lb.'s of force.

II. Adding vectors.

When we *add* vectors we always do it tip to tail (sketch). And the opposite of a vector just reverses it's direction 180, so subtracting is just adding the opposite. Finally we can always multiply a vector by some number and it will shrink or grow proportionally.

The question we asked earlier and the one that will recur is how to determine the magnitude and direction of the resultant from adding vectors.

Let's consider an alternative way to represent vectors and this may simplify the effort. Since a vector is in essence a line segment in the coordinate plane beginning at the origin, it is determined by its end-point. Then converting the vector from direction and magnitude (sketch) into coordinates and back should be pretty easy. In fact, a lot like our opener.
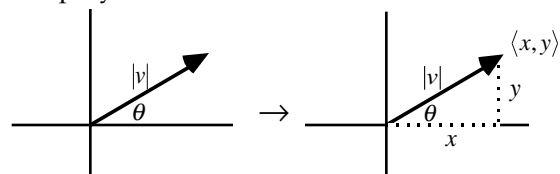
From the opener we saw a vector with magnitude 12 and direction 52° has coordinates (7.4, 9.5) or more specifically, given to us by $\left(12\cos 52°, 12\sin 52°\right)$. In order to tell you this isn't just a point but coordinates representing a vector, we write it <7.4, 9.5> (note the angled brackets should remind you of "V"'s for Vector). The conversion is pretty easy to see once you've done a few: $\langle |v|\cos\theta, |v|\sin\theta \rangle$ (point to sketch).

To convert back you notice that the coordinates give the vertical and horizontal components of the vector so this is a matter of finding r given the legs of a right triangle. The direction is similar (just keep track of the location of your reference Δ). This results in $r^2 = x^2 + y^2$ and $\theta = \arctan\left(\frac{y}{x}\right)$.

Why do we care? Because adding vectors with coordinates is ridiculously easy: $\langle a,b \rangle + \langle c,d \rangle = \langle a+c,b+d \rangle$

Then adding two vectors in the form with direction and magnitude is a matter of converting to coordinates, adding, and converting back.

| Example | Find $\vec{u}+\vec{v}$ if $|\vec{u}|=15$ with direction $\theta=67°$ and $|\vec{v}|=21$ with direction $\theta=128°$ (See Fig. 1).

$\vec{u}+\vec{v}=\langle 15\cos 67° + -21\cos 52°, 15\sin 67° + 21\sin 52° \rangle =$ <–7.06, 30.36> (See Fig. 2)

(Note reference angle used to emphasize how we work with different quadrants)

Then the magnitude is $\sqrt{(-7.06)^2+(30.36)^2} \approx 31.17$ and the direction is $\theta=\arctan\left(\frac{30.36}{-7.06}\right) \approx -77°$ (Fig. 3).
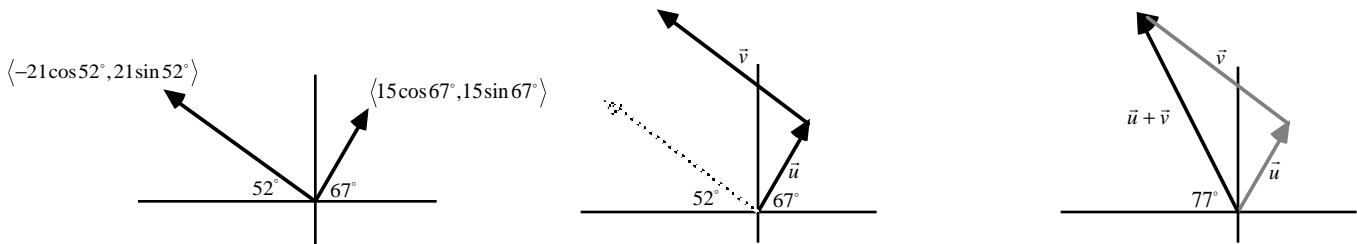
Note that $\theta \approx -77°$ tells us the angle occurs in quadrant II since arctangent outputs values between 0 and 180°.

Perhaps a more practical application of vector addition tells us what the resulting heading and velocity of an airplane would be when met with a cross wind.

| Example: |

An airplane is flying at 450 mph heading N40°E while a wind is blowing at 35 mph heading S70°E (or E20°S). Determine the resulting speed and direction of the plane.

If the wind were blowing in the same direction as the plane, we know it would add 35 mph to the speed of the plane. Likewise, in the opposite direction it subtracts 35 mph from the velocity of the plane. But like most cases, these velocities aren't in the same or opposite directions. The idea behind adding vectors is that we understand how forces behave when they are in the same or opposite directions. If we break a vector down into vertical and horizontal components then the horizontal vectors must either be in the same or opposite directions and likewise the vertical components.

This can be done with geometry and a lot of additional lines but we should at least agree that it is challenging. Adding vectors makes this much more approachable.

You should try to complete this example. The solution should conclude something like this:

$\langle 450\cos 50° + 35\cos 20°, 450\sin 50° + -35\sin 20° \rangle \approx \langle 322,333 \rangle$

So Magnitude 463 mph with heading N44°E ($\theta=46°$).