Assignment 2

This assignment is based on integration problems given during the Spring 2003 integration bee at Trinity College in Hartford Connecticut.

This assignment is worth 25 points. You will get 1 point for each correct solution. For each correct problem over 25, you will get 1 bonus point; thus you can get a maximum of 7 bonus points if you get all 32 problems correct.

Instructions:

- You will submit one page with the answers to the problems in addition to all the pages showing the worked out solutions.
- All work must be neat, completely correct and with perfect notation.
- Each integral must be evaluated using methods discussed in class (section 5.5 and sections 7.1 to 7.5). You can also use the results summarized in the table on page 484 of your textbook.
- Since you can check your answers by differentiation, you should not submit any wrong answers. Answers are graded right/wrong.
- Don't forget the arbitrary constant of integration in the antiderivatives.
- The assignment is due on Monday, November 16 at the beginning of class.

Spring 2003 Trinity Integration Bee

(1) $\int e^3\,dx$

(2) $\int \frac{x+\frac{x^2}{3}}{x}\,dx$

(3) $\int \sin^2 x + \cos^2 x\,dx$

(4) $\int \sin^2 x\,dx$

(5) $\int \ln x\,dx$

(6) $\int \frac{x}{\sqrt{1+x^2}}\,dx$

(7) $\int xe^x\,dx$

(8) $\int e^x \cos x\,dx$

(9) $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^2}$

(10) $\int \sqrt[3]{8x^7}\,dx$

(11) $\int \tan x\,dx$

(12) $\int 3^x x^2\,dx$

(13) $\int \frac{1}{x^2+2x}\,dx$

(14) $\int \frac{dx}{x - x\ln(x)}$

(15) $\int \frac{\sin\left(\frac{27}{x}\right)}{35x^2}\,dx$

(16) $\int \frac{e^{2x}}{e^{4x}+1}\,dx$

(17) $\int \cos(\sqrt{x})\,dx$

(18) $\int e^{e^x+x}\,dx$

(19) $\int x^2 \sin x\,dx$

(20) $\int 3x\sec^2 4x\,dx$

(21) $\int \frac{1}{\sqrt{4x^2+9}}\,dx$

(22) $\int \cos^3 x\,dx$

(23) $\int \frac{x^2+x+1}{x+1}\,dx$

(24) $\int \sec x\,dx$

(25) $\int \frac{\cos(x)}{2-\sin(x)}\,dx$

(26) $\int \tan^3 x \sec^5 x\,dx$

(27) $\int \frac{9x^2+12x+7}{9x^2+12x-4}\,dx$

(28) $\int \frac{2x+1}{(x+5)^{100}}\,dx$

(29) $\int (x+3)^{75}(x+2)\,dx$

(30) $\int \frac{\sqrt{1-\ln(x)^2}}{x}\,dx$

(31) $\int \frac{\sec^2 x}{(1+\tan x)^2}\,dx$

(32) $\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

1. $e^3 x + C$

2. $x + \frac{1}{6}x^2 + C$

3. $x + C$

4. $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$

5. $x\ln x - x + C$

6. $(1+x^2)^{1/2} + C$

7. $xe^x - e^x + C$

8. $\frac{1}{2}e^x(\sin x + \cos x) + C$

9. $\frac{-2}{1+\sqrt{x}} + C$

10. $\frac{3}{5}x^{10/3} + C$

11. $-\ln|\cos x| + C$

12. $3^x\left[\frac{x^2}{\ln 3} - \frac{2x}{(\ln 3)^2} + \frac{2}{(\ln 3)^3}\right] + C$

13. $\frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C$

14. $-\ln|1-\ln x| + C$

15. $\frac{1}{945}\cos\left(\frac{27}{x}\right) + C$

16. $\frac{1}{2}\tan^{-1}(e^{2x}) + C$

17. $2\left(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right) + C$

18. $e^{e^x} + C$

19. $-x^2\cos x + 2x\sin x + 2\cos x + C$

20. $\frac{3}{4}\left[x\tan 4x + \frac{1}{4}\ln|\cos 4x|\right] + C$

21. $\frac{1}{2}\ln\left|\frac{2}{3}x + \sqrt{1+\frac{4}{9}x^2}\right| + C$

22. $\sin x - \frac{1}{3}\sin^3 x + C$

23. $\frac{1}{2}x^2 + \ln|x+1| + C$

24. $\ln|\sec x + \tan x| + C$

25. $-\ln|2-\sin x| + C$

26. $\frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C$

27. $x - \frac{1}{3x+2} + C$

28. $-\frac{1}{49(x+5)^{98}} + \frac{1}{11(x+5)^{99}} + C$

29. $\frac{1}{77}(x+3)^{77} + \frac{1}{76}(x+3)^{76} + C$

30. $\frac{1}{2}\sin^{-1}(\ln x) + \frac{1}{2}\ln x\sqrt{1-(\ln x)^2} + C$

31. $-\frac{1}{1+\tan x} + C$

32. $-\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$

1. $\int e^3\,dx = \boxed{e^3x + C}$

2. $\int \frac{x + x^{2/3}}{x}\,dx = \int 1 + \frac{1}{3}x\,dx = \boxed{x + \frac{1}{6}x^2 + C}$

3. $\int \sin^2 x + \cos^2 x\,dx = \int 1\,dx = \boxed{x + C}$

4. $\int \sin^2 x\,dx = \int \frac{1}{2}(1 - \cos 2x)\,dx = \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$

$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin 2x + C}$

5. $\int \ln x\,dx$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$x\ln x - \int 1\,dx = x\ln x - x$

$\int \ln x\,dx = \boxed{x\ln x - x + C}$

6. $\int \frac{x}{\sqrt{1+x^2}}\,dx$ $\quad u = 1 + x^2$

$du = 2x\,dx \rightarrow x\,dx = \frac{1}{2}du$

$\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot 2u^{1/2} = u^{1/2} + C$

$\boxed{(1+x^2)^{1/2} + C}$

7. $\int xe^x\,dx$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$\int xe^x\,dx = xe^x - \int e^x\,dx = \boxed{xe^x - e^x + C}$

8. $\int e^x \cos x \, dx$

$u = e^x \qquad dv = \cos x \, dx$

$du = e^x \, dx \qquad v = \sin x$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

LET $\quad u = e^x \qquad dv = \sin x \, dx$

$du = e^x \, dx \qquad v = -\cos x$

$$\boxed{\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx}$$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\rightarrow \quad 2\int e^x \cos x \, dx = e^x (\sin x + \cos x)$$

$$\Rightarrow \quad \boxed{\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + c}$$

9. $\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})^2}$

$u = 1 + \sqrt{x} = 1 + x^{1/2}$

$du = \frac{1}{2} x^{-1/2} \, dx = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} \, dx \Rightarrow \frac{1}{\sqrt{x}} \, dx = 2 \, du$

$$2\int \frac{1}{u^2} \, du = 2\int u^{-2} \, du = 2 \cdot -\frac{1}{u} + C$$

$$\boxed{\frac{-2}{1+\sqrt{x}} + C}$$

10. $\int \sqrt[3]{8x^7} \, dx = \int 2 x^{7/3} \, dx = 2 \cdot \frac{3}{10} x^{10/3} + C = \boxed{\frac{3}{5} x^{10/3} + C}$

11. $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$ $\quad u = \cos x, \quad du = -\sin x \, dx$

$\sin x \, dx = -du$

$= -\int \frac{1}{u} \, du = -\ln|u| + c = -\ln|\cos x| + c$

12. $\int 3^x \cdot x^2 \, dx$

$u = x^2 \quad dv = 3^x \, dx$

$du = 2x \, dx \quad v = \frac{1}{\ln 3} \cdot 3^x$

$\frac{1}{\ln 3} \cdot x^2 3^x - \int \frac{1}{\ln 3} \cdot 2x \cdot 3^x \, dx = \frac{1}{\ln 3}\left[x^2 3^x - 2\int x \cdot 3^x \, dx\right]$

$\int x \cdot 3^x \, dx \qquad u = x \quad dv = 3^x \, dx$

$du = dx \quad v = \frac{1}{\ln 3} \cdot 3^x$

$= \frac{1}{\ln 3} x \cdot 3^x - \int \frac{1}{\ln 3} \cdot 3^x \, dx = \frac{1}{\ln 3} \cdot x \cdot 3^x - \frac{1}{(\ln 3)^2} \cdot 3^x$

$\frac{1}{\ln 3}\left[x^2 3^x - 2\left(\frac{1}{\ln 3} x 3^x - \frac{1}{(\ln 3)^2} 3^x\right)\right]$

OR $\frac{x^2 3^x}{\ln 3} - \frac{2x 3^x}{(\ln 3)^2} + \frac{2 \cdot 3^x}{(\ln 3)^3}$ OR $3^x\left[\frac{x^2}{\ln 3} - \frac{2x}{(\ln 3)^2} + \frac{2}{(\ln 3)^3}\right]$

DIFF.

$3^x\left[\frac{2x}{\ln 3} - \frac{2}{(\ln 3)^2}\right] + 3^x \ln 3\left[\frac{x^2}{\ln 3} - \frac{2x}{(\ln 3)^2} + \frac{2}{(\ln 3)^3}\right]$

$3^x\left[\frac{2x}{\ln 3} - \frac{2}{(\ln 3)^2}\right] + 3^x\left[x^2 - \frac{2x}{(\ln 3)^2} + \frac{2}{(\ln 3)^2}\right]$

$= 3^x \cdot x^2$

13. $\int \frac{1}{x^2+2x}\,dx = \int \frac{1}{x(x+2)}\,dx$

$\frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow A(x+2)+Bx = 1$

$\Rightarrow A = \frac{1}{2},\quad B = -\frac{1}{2}$

$\frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2|$

DIFF. $\frac{1}{2}\cdot\frac{1}{x} - \frac{1}{2}\cdot\frac{1}{x+2} = \frac{x+2}{2x(x+2)} - \frac{x}{2x(x+2)} = \frac{2}{2x(x+2)} = \frac{1}{x(x+2)}$

14. $\int \frac{dx}{x - x\ln x} = \int \frac{dx}{x(1-\ln x)}$ $\quad u = 1-\ln x$

$du = -\frac{1}{x}\,dx \Rightarrow \frac{1}{x}\,dx = -du$

$-\int \frac{du}{u} = -\ln|u| = -\ln|1-\ln x| + C$

15. $\int \frac{\sin\left(\frac{27}{x}\right)}{35x^2}\,dx = \int \frac{\sin(27x^{-1})}{35x^2}\,dx$ $\quad u = 27x^{-1}$

$du = -27x^{-2}\,dx$

$\Rightarrow x^{-2}dx = -\frac{1}{27}du$

$-\frac{1}{35}\cdot\frac{1}{27}\int \sin u\,du = +\frac{1}{945}\cos u + C$

$\frac{1}{945}\cos\left(\frac{27}{x}\right) + C$

16. $\int \frac{e^{2x}}{e^{4x}+1}\,dx$ $\quad u = e^{2x}, \quad du = 2e^{2x}\,dx = 2u\,dx$

$u^2 = e^{4x} \quad \Rightarrow dx = \frac{du}{2u}$

$\int \frac{\cancel{u}}{u^2+1} \cdot \frac{du}{\cancel{2u}} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1}(u) + C$

$\frac{1}{2}\tan^{-1}(e^{2x}) + C$

DIFF. $\frac{1}{2}\cdot\left(\frac{1}{1+e^{4x}}\right)\cdot 2e^{2x}$

---

17. $\int \cos\sqrt{x}\,dx \qquad u = \sqrt{x} = x^{1/2} \qquad du = \frac{1}{2}x^{-1/2}dx = \frac{1}{2\sqrt{x}}\,dx$

$du = \frac{1}{2u}\,dx$

$\Rightarrow dx = 2u\,du$

$\int 2u\cos u\,du = 2\int u\cos u\,du$

$w = u \quad dv = \cos u\,du$

$dw = du \quad v = \sin u$

$u\sin u - \int \sin u\,du = u\sin u + \cos u$

$2(u\sin u + \cos u) + C \quad$ OR $\quad 2(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}) + C$

DIFF. $2\left[\cancel{\sqrt{x}}\cdot\cos\sqrt{x}\cdot\frac{1}{2}\cdot\frac{1}{\cancel{\sqrt{x}}} + \sin\sqrt{x}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x}} + -\sin\sqrt{x}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\right]$

$\cos\sqrt{x}$

18. $\int e^{e^x+x}\,dx = \int e^{e^x}\cdot e^x\,dx, \quad u=e^x \quad du = e^x\,dx$

$\int e^u\,du = e^0 + C, \quad e^{e^x}+C$

19. $\int x^2 \sin x\,dx$

$u=x^2 \quad dv=\sin x\,dx$

$du=2x\,dx \quad v=-\cos x$

$-x^2\cos x + \int 2x\cos x\,dx = -x^2\cos x + 2\int x\cos x\,dx$

$u=x \quad dv=\cos x\,dx$

$du=dx \quad v=\sin x$

$x\sin x - \int \sin x\,dx$

$x\sin x + \cos x$

$-x^2\cos x + 2(x\sin x + \cos x) + C$

DIFF.

$-x^2(-\sin x) + \cos x(-2x) + 2x\cos x + 2\sin x + 2(-\sin x)$

20. $\int 3x\sec^2 4x\,dx \quad u=3x \quad dv=\sec^2 4x\,dx$

$du=3dx \quad v=\frac{1}{4}\tan 4x$

$\frac{3}{4}x\tan 4x - \frac{3}{4}\int \tan 4x\,dx$

$\int \tan 4x\,dx = \int \frac{\sin 4x}{\cos 4x}\,dx \quad u=\cos 4x$

$du = -4\sin 4x\,dx$

$\Rightarrow \sin 4x\,dx = -\frac{1}{4}\,du$

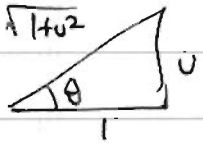
$-\frac{1}{4}\int \frac{du}{u} = -\frac{1}{4}\ln|u|$

$\frac{1}{\sec\theta+\tan\theta} \cdot \sec\theta\tan\theta+\sec^2\theta$

21. $\int \frac{1}{\sqrt{4x^2+9}}\,dx$ $\quad x = \frac{3}{2}u$ $\quad dx = \frac{3}{2}du$

$x^2 = \frac{9}{4}u^2$

$\int \frac{1}{\sqrt{9u^2+9}} \cdot \frac{3}{2}du = \frac{3}{2}\cdot\frac{1}{3}\int \frac{1}{\sqrt{u^2+1}}\,du$



$\frac{1}{2}\int \frac{1}{\sqrt{u^2+1}}\,du$ $\quad u = \tan\theta$ $\quad du = \sec^2\theta\,d\theta$

$\frac{1}{2}\int \frac{1}{\sec\theta}\cdot\sec^2\theta\,d\theta = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta|+c$

$\frac{1}{2}\ln\left|\sqrt{1+u^2}+u\right| = \frac{1}{2}\ln\left|\sqrt{1+\left(\frac{2}{3}x\right)^2}+\frac{2}{3}x\right|$

$\frac{1}{2}\ln\left|\frac{2}{3}x+\sqrt{1+\frac{4}{9}x^2}\right|$

22. $\int \cos^3x\,dx = \int \cos x\cdot(1-\sin^2x)\,dx$ $\quad u = \sin x$ $\quad du = \cos x\,dx$

$\int 1-u^2\,du =$ $\quad u - \frac{1}{3}u^3+c$ $\quad \sin x - \frac{1}{3}\sin^3x+c$

23. $\int \frac{x^2+x+1}{x+1}\,dx = 2\cdot\int \frac{x^2+x+1}{2x+2}\,dx = 2\int \frac{x^2+x+1}{2x+1+1}\,dx$

$u = x^2+x+1$
$du = 2x+1\,dx$

23. $\int \frac{x^2+x+1}{x+1}\,dx$

$$x+1 \;\overline{)\; x^2+x+1 \;}\;\; x$$
$$-x^2-x$$
$$1$$

$$\int x + \frac{1}{x+1}\,dx$$

$$\frac{1}{2}x^2 + \ln|x+1| + C$$

24. $\int \sec x = \ln|\sec x + \tan x|$

$$\frac{d}{dx}\ln|\sec x+\tan x| = \frac{1}{\sec x+\tan x}\cdot \sec x\tan x + \sec^2 x$$

$$= \frac{\sec x(\tan x+\sec x)}{\sec x+\tan x} = \sec x.$$

25. $\int \frac{\cos x}{2-\sin x}\,dx$ $\qquad u = 2-\sin x \qquad du = -\cos x\,dx$

$$\longrightarrow \quad \cos x\,dx = -du$$

$$\int -\frac{du}{u} = -\ln|u| + C$$
$$-\ln|2-\sin x| + C$$

26. $\int \tan^3 x \sec^5 x\,dx$ $\qquad \sin^2 x+\cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$

$$\int \tan^3 x \sec x \sec^4 x\,dx$$
$$\int \sec x \tan x\,(\sec^2 x - 1)\sec^4 x\,dx \qquad u = \sec x \quad du = \sec x\tan x\,dx$$

$$\int (u^2-1)u^4\,du = \int u^6 - u^4\,du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$$

$$\frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C$$

27. $\int \frac{9x^2+12x+7}{9x^2+12x+4}\,dx$

$$9x^2+12x+4 \;\overline{)\;9x^2+12x+7}$$
$$-\,9x^2+12x+4$$
$$\overline{\qquad\qquad 3}$$

$$\int 1 + \frac{3}{9x^2+12x+4}\,dx = x + 3\int \frac{1}{(3x+2)^2}\,dx$$

$u = 3x+2 \quad du = 3\,dx \Rightarrow dx = \frac{1}{3}\,du$

$$x + 3\cdot\frac{1}{3}\int u^{-2}\,du = x + \left(-\frac{1}{u}\right) = x - \frac{1}{3x+2} + C$$

28. $\int \frac{2x+1}{(x+5)^{100}}\,dx$

$u = x+5 \quad du = dx$

$x = u-5 \Rightarrow 2x+1 = 2u-9$

$$\int \frac{2u-9}{u^{100}}\,du = \int 2u^{-99} - 9u^{-100}\,du = -\frac{2}{98}u^{-98} - 9\cdot\frac{1}{99}u^{-99} + C$$

$$-\frac{1}{49u^{98}} + \frac{1}{11u^{99}} + C$$

29. $\int (x+3)^{75}(x+2)\,dx$

$u = x+3 \Rightarrow x = u-3 \Rightarrow x+2 = u-1$

$du = dx$

$$\int u^{75}(u-1)\,du = \int u^{76} - u^{75}\,du = \frac{1}{77}u^{77} - \frac{1}{76}u^{76}$$

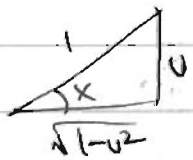
30. $\int \frac{\sqrt{1-(\ln x)^2}}{x}\,dx$ $\quad u = \ln x, \quad du = \frac{1}{x}\,dx$

$\int \sqrt{1-u^2}\,du$ $\quad u = \sin\theta \quad du = \cos\theta\,d\theta$

$1-u^2 = 1-\sin^2\theta = \cos^2\theta$

$\int \cos\theta \cdot \cos\theta\,d\theta = \int \cos^2\theta\,d\theta = \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$

$\frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) = \frac{1}{2}(\theta + \sin\theta\cos\theta)$



$\frac{1}{2}(\theta + \quad)$

$\frac{1}{2}\left(\sin^{-1}u + u\sqrt{1-u^2}\right) = \frac{1}{2}\left(\sin^{-1}(\ln x) + \ln x\sqrt{1-(\ln x)^2}\right)$

31. $\int \frac{\sec^2 x}{(1+\tan x)^2}\,dx$ $\quad u = 1+\tan x \quad du = \sec^2 x\,dx$

$\int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{1+\tan x} + C$

32. $\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$ $\quad x = \sin\theta \quad dx = \cos\theta\,d\theta$

$(1-x^2)^{3/2} = (\cos^2\theta)^{3/2} = \cos^3\theta$

$x^6 = \sin^6\theta$

$\int \frac{\cos^4\theta}{\sin^6\theta}\,d\theta$

$\int \frac{(1-x^2)^{3/2}}{(x^4)^{3/2}}\,dx = \int \frac{(1-x^2)^{3/2}}{x^4}\,dx = \int (x^{-4}+x^{-2})^{3/2}\,dx$

$\int \frac{x^4}{(1-x^2)^{3/2}}\,dx$

#32.

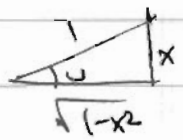
$$\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$$

$u = \arcsin x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx \Rightarrow dx = \sqrt{1-x^2}\,du$

$x = \sin u \Rightarrow 1-x^2 = 1-\sin^2 u = \cos^2 u$

$x^6 = \sin^6 u$



$$\int \frac{(\cos^2 u)^{3/2}}{\sin^6 u}\cos u\,du = \int \frac{\cos^4 u}{\sin^6 u}\csc^6 u\,du$$

$$\int \frac{\cos^4 u}{\sin^6 u}\,du = \int \cot^4 u \csc^2 u\,du$$

$v = \cot u$

$dv = -\csc^2 u\,du$

$$= -\int v^4\,dv = -\frac{1}{5}v^5 + C$$

$$= -\frac{1}{5}\cot^5 u + C = -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$$