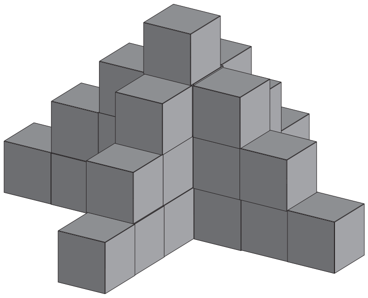
Series Problems Name: **Solutions**

$\boxed{1.}$ Consider the pyramid of bricks shown to the right.

(a) Write a formula for the number of bricks on the n^{th} level.

Solution: Linear Table:.

n	1	2	3	4	...	n
S_n	1	5	9	13	...	$4n-3$



(b) Find the total number of bricks needed to build this pyramid 100 rows high (don't just find the number for the 100^{th} level).

Solution: Quadratic again, note the rate of change in the rate of change is constant. We can factor the outputs for a clue, use pre-calc mehtods or TI-83 regression.

n	1	2	3	4	...	n
S_n	1	6	15	28	...	$n(2n-1)$

$\longrightarrow 100(199) = 19{,}900$

$\boxed{2.}$ Determine if the sequence converges and determine the limit of each convergent sequence.

(a) $f(n) = \dfrac{n+1}{n} - \dfrac{n}{n+1}$

Solution: $\dfrac{n+1}{n} - \dfrac{n}{n+1} = \dfrac{2n+1}{n(n+1)}$

$$\lim_{n\to\infty} \frac{2n+1}{n(n+1)} = 0$$

(b) $f(n) = \dfrac{1}{\sqrt[n]{2}}$

Solution: $\lim\limits_{n\to\infty} 2^{-1/n} = 1$

(c) $f(n) = \left(1+\dfrac{2}{n}\right)^n$

Solution: $\lim\limits_{n\to\infty} \left(1+\dfrac{2}{n}\right)^n = e^2$

(d) $f(n) = (-1)^n\left(\dfrac{9}{10}\right)^n$

Solution: $\lim\limits_{n\to\infty} (-1)^n\left(\dfrac{9}{10}\right)^n = 0$

$\boxed{3.}$ A superball rebounds to 70% of its previous height every time it bounces. If it is dropped initially from a height of 6 feet, what is the total vertical distance it travels as it bounces up and down?

Solution: $6 + 2\cdot 6(0.7)^1 + 2\cdot 6(0.7)^2 + 2\cdot 6(0.7)^3 + \cdots = 6 + \sum\limits_{n=1}^{\infty} 12(0.7)^n = 6 + \dfrac{12}{1-0.7} = 46$ feet.

$\boxed{4.}$ Write 5.4321321321. . . as a fraction.

Solution: $5.4 + \dfrac{0.0321}{1-0.001} = \dfrac{54}{10} + \dfrac{321}{9990} = \dfrac{18089}{3330}$

$\boxed{5.}$ Show $\sum\limits_{n=1}^{\infty} \dfrac{1}{(2n-1)(2n+1)} = \dfrac{1}{2}$.

Solution: $\sum\limits_{n=1}^{\infty} \dfrac{1}{(2n-1)(2n+1)} = \sum\limits_{n=1}^{\infty} \dfrac{1}{2(2n-1)} - \dfrac{1}{2(2n+1)} =<$ Telescoping Series $>= \dfrac{1}{2} - \lim\limits_{n\to\infty} \dfrac{1}{2(2n+1)} = \dfrac{1}{2}$

$\boxed{6.}$ Provide an example that shows if $\sum a_n$ diverges and $\sum b_n$ diverges, $\sum a_n - b_n$ *does not* diverge.

Solution: e.g. $a_n = b_n = \frac{1}{n}$.

$\boxed{7.}$ Determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$

Sol: $\displaystyle\int_0^{\infty} \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x = 2\sqrt{x}(\ln(x) - 2)\Big|_0^{\infty}$
Diverges, therefore the series diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$

Sol: $\displaystyle\int_0^{\infty} \frac{3}{\sqrt{x}(\sqrt{x}+1)}\,\mathrm{d}x = 3(\sqrt{x}+1)^2\Big|_0^{\infty}$
Diverges, therefore the series diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(\ln n)}$

Sol: $\ln(\ln n) < \ln n < n$ so
$\displaystyle\frac{1}{\ln(\ln n)} > \frac{1}{n}.\quad \sum \frac{1}{n}$ Diverges
So (c) diverges by comparison.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\ln n}$

Sol: $\sqrt{n}\ln n < \sqrt{n}\sqrt{n} = n$ so
$\displaystyle\frac{1}{\sqrt{n}\ln n} > \frac{1}{n}.\quad \sum \frac{1}{n}$ Diverges
So (d) diverges by comparison.

(e) $\displaystyle\sum_{n=1}^{\infty} n!\, e^{-n} = \sum_{n=1}^{\infty} \frac{n!}{e^n}$

Sol: Diverges by Ratio test.

$\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$

Sol: Diverges by Limit Comparison test with $\sum \frac{1}{n}$.

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)}{n+1}$

Sol: Diverges. Fails A/S test.
$\lim_{n\to\infty} \frac{n-1}{n+1} \neq 0$.

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{2}}$

Sol: Diverges.
$\lim_{n\to\infty} \frac{1}{\sqrt[n]{2}} \neq 0$.

(i) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{9}{10}\right)^n$

Sol: Converges - Geometric, $r < 1$.
$S = -\frac{9}{19}$.

$\boxed{8.}$ Find the Taylor Series, centered on $x = 0$, for $f(x) = \dfrac{1}{1-x^2}$.
Then determine which values of x it converges for.

Solution: Use geomstric series with $x \longrightarrow x^2$: $\displaystyle\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots = \sum_{n=0}^{\infty} x^{2n}$

From the Ratio Test we have $\displaystyle\lim_{n\to\infty} \frac{\left|x^{2(n+1)}\right|}{\left|x^{2n}\right|} = x^2$ so the series converges for $x^2 < 1$ or $-1 < x < 1$. Note that substituting $n = -1$ and $n = 1$ into the original series prodcues the divergent series $1 + 1 + 1 + \cdots$.

$\boxed{9.}$ Find the Taylor Series, centered on $x = 0$, for $f(x) = \dfrac{1}{(1-x^2)^3}$.
Then determine which values of x it converges for.

Solution: Use Binomial Series with $p = -3$ and $x \longrightarrow (-x^2)$: $\displaystyle\frac{1}{(1-x^2)^3} = 1 + 3x^2 + \frac{12}{2!}x^4 + \frac{60}{3!}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(n+2)!}{2\cdot n!}x^{2n}$.

Equivalently, we have $\displaystyle\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2}x^{2n}$. From the Ratio Test we have $\displaystyle\lim_{n\to\infty} \frac{\left|\frac{(n+3)!x^{2(n+1)}}{2(n+1)!}\right|}{\left|\frac{x^{2n}(n+2)!}{2\cdot n!}\right|} = x^2 \lim_{n\to\infty} \frac{n+3}{n+1} = x^2$ so the series converges for $x^2 < 1$ or $-1 < x < 1$. Again we note that the weries diverges for $x = \pm 1$.

$\boxed{10.}$ The Taylor series for $\arctan x$ (centered at $x = 0$) is given below.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

We know that $\arctan(1) = \dfrac{\pi}{4}$ so using the series above we can approximate π ($\pi = 4\arctan(1)$).

Use the error bound for the series above to determine the degree of the Taylor Polynomial necessary to approximate π accurate to 6 decimal places.

Solution: We derive the Taylor Series for $\arctan x$ by integrating the geometric series $\dfrac{1}{1+x^2} = \dfrac{\mathrm{d}}{\mathrm{d}x}\arctan x$.
Recall that the geometric series has the form $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$. The error bound has the form $E_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$. Now for $f(x) = (1-x)^{-1}$ we have $f^{(n+1)}(x) = (-1)^{n+1}(n+1)!(1-x)^{-(n+2)}$ which is largest when $x = 0$ so we have $f^{(n+1)}(0) = (-1)^{n+1}(n+1)!$. The error bound is therefore determined by $E_n(x) = \frac{(n+1)!}{(n+1)!}x^{n+1} = x^{n+1}$.
The series for $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots$ therefore has an error bound $E_n(x) = (x^2)^{n+1} = x^{2n+2}$.
Integrating the series gives us the Taylor Series for $\arctan x$ so integrating the error bound gives us the error bound for $\arctan x$:
$E_n(x) = \displaystyle\int x^{2n+2}\,\mathrm{d}x = \frac{1}{2n+3}x^{2n+3}$. From this it follows that if we want the error approximating π to be less than 0.000001, we need $E_n(x) < \frac{0.000001}{4}$ at $x = 1$.
Solving $\dfrac{1}{2n+3}(1)^{2n+3} = \dfrac{0.000001}{4}$ gives us a ridiculously large value for n (about 2,000,000).

$\boxed{11.}$ Write a power series that has (1, 5) as its interval of convergence.

Solution: This interval is centered at $x = 3$ and has a radius of 2. Consider the Ratio Test. We want $\displaystyle\lim_{n\to\infty}\frac{|C_{n+1}(x-3)^{n+1}|}{|C_n(x-3)^n|} = \frac{1}{2}(x-3)$ (because we solve $\frac{1}{2}(x-3) < 1 \longrightarrow (x-3) < 2$).

Consider geometric (exponential) variation of power series: $\displaystyle\sum_{n=0}^{\infty}\frac{(x-3)^n}{2^n}$.

$\boxed{12.}$ Find the exact interval of convergence for the Taylor series of $f(x) = \ln(x^2 + x + 1)$ centered at $x = 0$.

Solution: We know the Taylor Series for $f(x) = \ln(x+1)$ centered at $x = 0$ is given by (1). Substituting $x^2 + x = x(x+1)$ for x gives the desired series:

$$\begin{align}
\ln(x+1) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \tag{1}\\
\ln(x^2+x+1) &= \ln(x(x+1)+1) \tag{2}\\
&= x(x+1) - \frac{[x(x+1)]^2}{2} + \frac{[x(x+1)]^3}{3} - \frac{[x(x+1)]^4}{4} + \cdots + (-1)^{n+1}\frac{[x(x+1)]^n}{n} + \cdots \tag{3}\\
&= x(x+1) - \frac{x^2(x+1)^2}{2} + \frac{x^3(x+1)^3}{3} - \frac{x^4(x+1)^4}{4} + \cdots + (-1)^{n+1}\frac{x^n(x+1)^n}{n} + \cdots \tag{4}
\end{align}$$

From the ratio test we have $\displaystyle\lim_{n\to\infty}\frac{\frac{|x^{n+1}(x+1)^{n+1}|}{n+1}}{\frac{|x^n(x+1)^n|}{n}} = |x(x+1)|\lim_{n\to\infty}\frac{n}{n+1} = |x(x+1)|$. Solving $|x(x+1)| < 1$ we have $-1 < x^2 + x < 1$. $x^2 + x$ is always greater than -1 (try QF) and solving $x^2 + x < 1$ gives us $\dfrac{-1-\sqrt{5}}{2} < x < \dfrac{-1+\sqrt{5}}{2}$ or about $-1.62 < x < 0.62$ for the interval of convergence.