Abstract

While often treated as a mathematical side show in introductory calculus texts, the hyperbolic functions $\sinh x$ and $\cosh x$ prove to be of vital importance in a variety of applications. In particular we shall examine the derivation of the formula used to model the *catenary*, or idealized hanging chain. The proof and subsequent applications require only general knowledge of calculus and elementary physics.

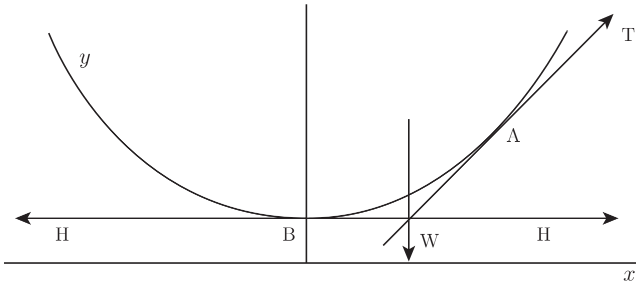


Figure 1: Catenary

Notes:

Catenary, from latin *catena* meaning chain.
This is one of those idealized models that never occurs in nature. In this case, an idealized chain or cable hanging from two points not in the same vertical line.

A steel catenary six miles long would break under its own weight.

Assumptions:

- End points not in same vertical line.
- Suspended from points in same horizontal plane (relaxed later).
- Uniform - as in constant shape and constant linear weight density, δ.
- Hangs symmetrically. Suppose otherwise - then vertex lies to one side but from reverse view, it would hang to the other side and we have $\sharp$.
  – Which suggests we assume symmetric equilibrium.

From the shape, the choice of y-axis as line of symmetry is obvious.
For the horizontal axis we will leave the curve above the x-axis. This choice will make more sense later.
We now add to our previous assumptions something more thought provoking.

- Tension is tangential at every point.

To see this first imagine cable reeled off from a spool. Given thorough (ideal) flexibility this will form a tangent.

Second, consider the special case of the point B at the bottom of the catenary. Let H be the horizontal force on B and V the vertical force on B. Then assuming equilibrium, H cancels and V is balanced by the weight of the point – infinitesimally small – which must be 0. $\therefore$ The only force at B is the horizontal force H which again is tangential to the curve. To cement this, imagine cutting the catenary at B - what direction must you pull in order to keep equilibrium?

Development:

Pick an arbitrary point, $A(x, y)$ on the catenary and consider the equilibrium state of AB.
The equilibrium of AB is composed of the following forces:

- H – the horizontal force at B.
- T – the tension of the catenary - tangential at $A(x, y)$.
- W – the weight of the section AB.

The vector combination of T and H must be equal and opposite to the weight of the section AB. (*The lines of force are concurrent in the center of gravity of* AB). Equivalently, the tension is exactly opposite the vector sum of the horizontal force, H, and the vertical force, W. The critical point here is the assumption of equilibrium and the observation that H is constant everywhere since it defines the horizontal pull of the chain at B. The left half of the chain puts up this constant pull, H, to the left.

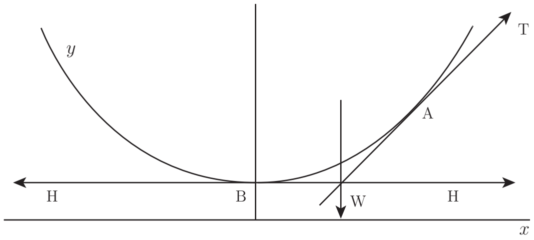


Figure 2: Forces

The Math:

We want to find a formula that describes the catenary in terms of x and y. We know the relationship between the forces at $A(x, y)$ and by representing these forces in terms of x and y, we should find a formula (or differential equation).
To begin with, W, the weight of AB, is given by the product of its length and its lineal density: $W = \delta \cdot s$, where s is the arclength of AB and δ the lineal weight density.
From the vector diagram below, we have $\tan\theta = \frac{W}{H} = \frac{\delta \cdot s}{H}$ and since the tangent at A has slope $\frac{\Delta y}{\Delta x}$, it follows that $\frac{dy}{dx} = \tan\theta = \frac{\delta \cdot s}{H}$

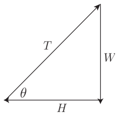


Figure 3: Vectors

The next step is to find s in terms of x and y:
This amounts to deriving the arclength formula which we will outline below.

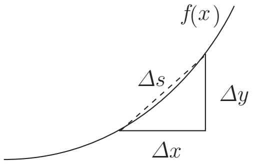


Figure 4: Arclength

$(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2$ so $\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$ and dividing by Δx we get $\frac{\Delta s}{\Delta x} \approx \sqrt{1 + (\frac{\Delta y}{\Delta x})^2}$. This is typically where we fake it a bit and take the limit as Δx goes to 0 and pass to the limit: $\frac{ds}{dx} = \sqrt{1 + (\frac{dy}{dx})^2}$ which you may recall as the arclength formula.

From our earlier discussion, we have $\frac{dy}{dx} = \frac{\delta \cdot s}{H}$ so now $\frac{d^2y}{dx^2} = \frac{\delta}{H}\frac{ds}{dx}$ and substituting, we obtain

$$\frac{d^2y}{dx^2} = \frac{\delta}{H}\sqrt{1 + (\frac{dy}{dx})^2}$$

We now have a second order differential equation to solve. Fortunately this equation lends itself to substitution so let $z = \frac{dy}{dx}$ and it follows $\frac{dz}{dx} = \frac{\delta}{H}\sqrt{1+z^2}$. For brevity let us also make the substitution $\frac{1}{c} = \frac{\delta}{H}$. Then separating variables, we have,

$$\frac{dz}{\sqrt{1+z^2}} = \frac{dx}{c} \Rightarrow \int \frac{dz}{\sqrt{1+z^2}} = \int \frac{dx}{c}$$

Applying the table of integrals or since $z = \frac{dy}{dx} = \tan\theta$ it follows $dz = \sec^2\theta$
Therefore $\int \frac{dz}{\sqrt{1+z^2}} = \int \sec\theta \, d\theta$ and thus $\ln \mid \sec\theta + \tan\theta \mid = \frac{x}{c} + k$.
Substituting back in, $\ln\left|\sqrt{1+z^2} + z\right| = \frac{x}{c} + k$.
and $\ln\left|\sqrt{1 + (\frac{dy}{dx})^2} + (\frac{dy}{dx})\right| = \frac{x}{c} + k$. Since $\frac{dy}{dx} = 0$ when $x = 0 \Rightarrow k = 0$.

Taking exp. of both sides: $e^{x/c} = \sqrt{1 + (\frac{dy}{dx})^2} + (\frac{dy}{dx})$ (note, abs. val. can be dispensed with since the root is always larger and positive). Squaring gives: $(e^{x/c} - \frac{dy}{dx})^2 = 1 + \left(\frac{dy}{dx}\right)^2$

$e^{2x/c} - 2\frac{dy}{dx}e^{x/c} = 1$

$\frac{e^{2x/c}-1}{2e^{x/c}} = \frac{dy}{dx}$

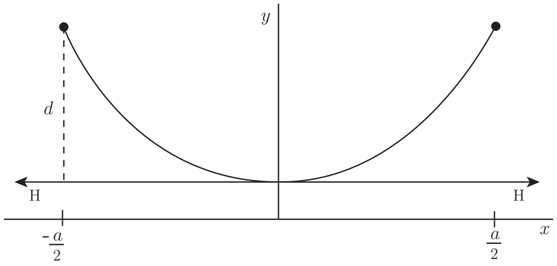
$\frac{e^{x/c}-e^{-x/c}}{2} = \frac{dy}{dx}$

$\sinh\frac{x}{c} = \frac{dy}{dx} \Rightarrow c \cdot \cosh\frac{x}{c} = y$, where we assume $y = c$ for $x = 0$.

We conclude that the equation for a catenary is given by:

$$y = c \cdot \cosh\left(\frac{x}{c}\right)$$

Additional Notes:



1. If we relax the requirement that the endpoints lie in the same horizontal plane, we note that H is constant at the lowest point, again, and this is the only critical requirement in our assumptions.
2. The length of the chain or cable can be found from:
   $\frac{l}{2} = \int_0^{a/2} \sqrt{1 + \sinh^2 \frac{x}{c}}\, dx$ and therefore $l = 2c \cdot \sinh \frac{a}{2c}$.
3. Finding the sag, d.
   $(l/2)^2 + c^2 = c^2 \sinh^2 \frac{a}{2c} + c^2 = c^2 \cosh^2 \frac{a}{2c} = (c+d)^2$. Where $c = \frac{H}{\delta}$ and $a/2$ is the x coordinate of the suspension point.
4. Tension:
   $T^2 \sin^2 \theta + T^2 \cos^2 \theta = T^2$. But $T \sin \theta = W$ and $T \cos \theta = H$ so we have $(\delta \cdot s)^2 + H^2 = T^2 \Rightarrow \delta^2 (l/2)^2 + \delta^2 c^2 = T^2$ so $T = \delta \sqrt{(l/2)^2 + c^2}$.
5. From the calculus of variations, the solution to the problem minimizing $\int_{x_0}^{x_1} 2\pi f(x) \sqrt{1 + [f'(x)]^2}$ is a catenary. That is, of all the curves lying entirely above the x-axis, when rotated about the x-axis, the catenary provides the surface of revolution with the least area.
6. The bulk of section I and many other brilliant insights can be found in Polya's *Mathematical Methods in Science*, an MAA publication.