

Show all relevant work!

1. Suppose  $h(x) = (f(x))^3$ . If  $f(1) = -2$  and  $f'(1) = 5$ , find  $h'(1)$ .

**Solution:**  $h'(x) = 3(f(x))^2 \cdot f'(x)$  so  $h'(1) = 3(f(1))^2 \cdot f'(1) = 3(-2)^2 \cdot 5 = 60$

2. The half-life of Pu-238 is 88 years. Let  $Q_0$  represent the initial quantity of Pu-238 and assume the decay of the element is continuous.

(a) Using base  $e$ , write the particular equation giving the quantity,  $Q$ , in grams as a function of time,  $t$ , in years.

**Solution:** We begin with  $Q(t) = Q_0 e^{kt}$ . Since half of  $Q_0$  remains after 88 years we write  $\frac{1}{2}Q_0 = Q_0 e^{88k}$ . It follows that  $\frac{1}{2} = e^{88k}$  so  $k = \frac{1}{88} \ln\left(\frac{1}{2}\right) \approx -0.007877$ . Therefore  $Q(t) = Q_0 e^{-0.007877t}$

(b) Find  $\frac{dQ}{dt}$  and explain what it represents. Include units in your explanation.

**Solution:**  $\frac{dQ}{dt} = -0.007877Q_0 e^{-0.007877t}$  grams/year. This is the rate at which the quantity (in grams) of Pu-238 is decaying at any time,  $t$ .

(c) What is the continuous rate of decay from your model in (a)? Compare and contrast this rate with your answer to part (b).

**Solution:**  $e^{-0.007877} \approx 0.9922$  so the annual rate of decay is roughly  $1 - 0.9922 = 0.0078$  or about 0.78% per year. The derivative with respect to time takes the percentage of the quantity present at time  $t$  and gives the decay rate in terms of actual quantity lost per year. The annual percent gives the relative change in quantity.

3. The daily cost,  $C$ , of running an air conditioner in Arizona depends on the temperature,  $H$ , as shown in the first table. The temperature in turn increases with the time of day,  $t$ , as shown in the second table. Determine the rate at which cost changes with time when  $t = 10$  and interpret the result.

$H$ (in $F^\circ$ )	90	95	100	105	110
$C(H)$ (\$)	4	4.75	6	7.50	9.15

$t$ (in hours past 00:00)	6	8	10	12	14
$H(t)$ (in $F^\circ$ )	90	97	100	112	119

**Solution:** We want to find  $\frac{d}{dt}C(H(t))$  at  $t = 10$ . From the Chain Rule  $\frac{d}{dt}C(H(t)) = C'(H(t)) \cdot H'(t)$  and our solution follows from  $C'(H(10)) \cdot H'(10)$ .

$$C'(H(10)) = C'(100) \approx \frac{7.50 - 4.75}{105 - 95} = 0.275. \quad H'(10) \approx \frac{112 - 97}{12 - 8} = 3.75$$

$$\text{Therefore } C'(H(10)) \cdot H'(10) \approx (0.275)(3.75) \approx 1.03.$$

So at 10:00 am, the cost of running an air conditioner in Arizona is increasing at a rate of \$1.03 per hour.

4. Suppose  $m'(x) = \frac{x^3}{\sqrt{1-x^4}}$ . What is a possible formula for  $m(x)$ ?

**Solution:** From our experience with the derivative of  $y = \sqrt{x}$ , we might guess that our solution is something like  $y = \sqrt{1-x^4}$ . From this we get  $y' = \frac{1}{2} \cdot \frac{-4x^3}{\sqrt{1-x^4}} = \frac{-2x^3}{\sqrt{1-x^4}}$ . Since we want  $m'(x) = \frac{x^3}{\sqrt{1-x^4}}$ , we need to eliminate the factor of  $-2$ . So if we go back to our original guess of  $y = \sqrt{1-x^4}$  and append the multiplicative inverse of  $-2$ , we get  $m(x) = \frac{-1}{2}\sqrt{1-x^4}$ .

5. Show that  $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$

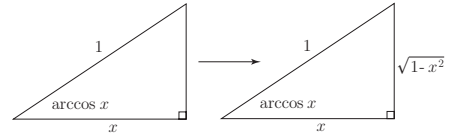
**Solution:** As with other inverse functions we begin by writing  $\cos(\arccos x) = x$  and differentiating both sides gives

$$\frac{d}{dx} \cos(\arccos x) = \frac{d}{dx} x \longrightarrow$$

$$-\sin(\arccos x) \cdot \frac{d}{dx} \arccos x = 1$$

$$\frac{d}{dx} \arccos x = \frac{1}{-\sin(\arccos x)}$$
 and from the diagram,

$$\text{this yields } \frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}.$$



6. Use the table below to help you find  $\frac{d}{dx} f^{-1}(x)$  evaluated at  $x = 4$ .

$x$	0	1	2	3	4
$f(x)$	1	2	4	5	5.6
$f'(x)$	1	$\frac{9}{5}$	$\frac{3}{2}$	$\frac{4}{5}$	$\frac{1}{3}$

**Solution:** In general we find the derivative function for  $f^{-1}(x)$  as follows:

$$f(f^{-1}(x)) = x \longrightarrow \frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx} x$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

From this we evaluate  $\frac{d}{dx} f^{-1}(x)$  at  $x = 4$ :

$$\left. \frac{d}{dx} f^{-1}(x) \right|_{x=4} = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{3/2} = \frac{2}{3}$$

7. Show  $\frac{d}{dx} b^x = \ln(b)b^x$ .

**Solution:** Since we know  $\frac{d}{dx} e^{kx} = ke^{kx}$  it follows that since  $b^x = (e^{\ln b})^x$  we have

$$\frac{d}{dx} b^x = \frac{d}{dx} e^{(\ln b)x} = e^{(\ln b)x} \cdot \ln b = b^x \cdot \ln b.$$

8. Find  $\frac{d}{dx} \log_b x$ .

**Solution:** As with most proofs involving inverse functions we begin with  $b^{\log_b x} = x$ .

$$\text{Taking derivative of both sides produces } \frac{d}{dx} b^{\log_b x} = \frac{d}{dx} x$$

$$\Rightarrow (\ln b)b^{\log_b x} \cdot \frac{d}{dx} \log_b x = 1$$

$$\Rightarrow (\ln b)x \cdot \frac{d}{dx} \log_b x = 1$$

$$\Rightarrow \frac{d}{dx} \log_b x = \frac{1}{(\ln b)x}$$

Note the similarity between  $\frac{d}{dx} \ln x = \frac{1}{x}$  and  $\frac{d}{dx} \log_b x = \frac{1}{(\ln b)x}$ . As with the derivative of exponential functions, they differ only by a factor of  $\ln b$  (or  $\frac{1}{\ln b}$  in this case).

9. Where is  $y = \arctan(1 - x^2)$  increasing and where is it decreasing? Where is it concave up?

**Solution:** From the Chain Rule we have  $f'(x) = \frac{-2x}{1+(1-x^2)^2}$ . The only place  $f'(x) = 0$  is at  $x = 0$ .

A quick check of values below and above 0 shows that  $f'(-1) > 0$  while  $f'(1) < 0$ .

Therefore  $f(x)$  is increasing on  $(-\infty, 0]$  and decreasing on  $[0, \infty)$ .

From the quotient Rule and Chain Rule we get

$$f''(x) = \frac{-2(1 + (1 - x^2)^2) - (-2x) \cdot 2(1 - x^2)(-2x)}{[1 + (1 - x^2)^2]^2} = \frac{6x^4 - 4x^2 - 4}{[1 + (1 - x^2)^2]^2} = \frac{2(3x^4 - 2x^2 - 2)}{[1 + (1 - x^2)^2]^2}$$

Setting the numerator equal to zero gives us  $2(3x^4 - 2x^2 - 2) = 0$ . Ignoring the 2 and letting  $u = x^2$  produces  $3u^2 - 2u - 2 = 0$  and the QF yields  $u = \frac{1+\sqrt{7}}{3}$ ; therefore  $x = \pm \sqrt{\frac{1+\sqrt{7}}{3}}$ . Testing yields:

$$\begin{array}{c|c|c} x < -\sqrt{\frac{1+\sqrt{7}}{3}} & -\sqrt{\frac{1+\sqrt{7}}{3}} < x < \sqrt{\frac{1+\sqrt{7}}{3}} & x > \sqrt{\frac{1+\sqrt{7}}{3}} \\ \hline f''(-2) > 0 & f''(0) < 0 & f''(2) > 0 \end{array}$$

It follows that  $y = \arctan(1 - x^2)$  is concave up on (roughly)  $(-\infty, -1.102)$  and  $(1.102, \infty)$  and concave down on  $(-1.102, 1.102)$ .