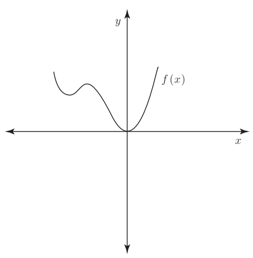
# Some Investigations

## 0.1 Shifts and Stretches

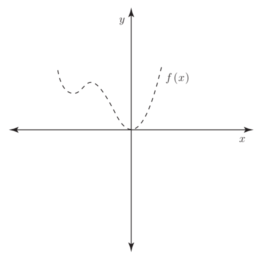
Recall the various manipulations of a function and their related effects. Complete the sketches and tables for each manipulation.

**Shifts** (vertical)

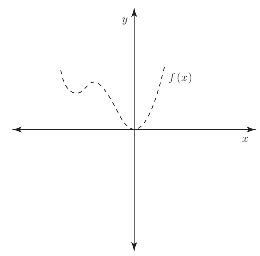
graphic:




$f(x)$




$f(x)+a$
$(a>0)$




$f(x)-a$
$(a>0)$

tabular:

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | 4 | 2 | 0 | 2 | 5 |

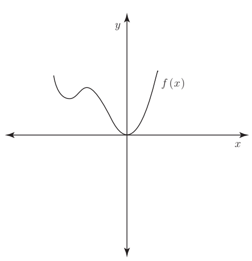
| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)+a$ | | | | | |

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)-a$ | | | | | |

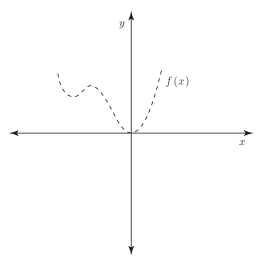
(e.g. $a=2$)

**Shifts** (horizontal)

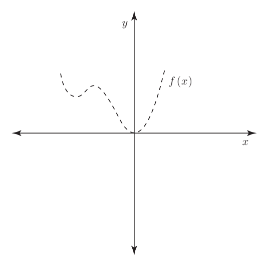
graphic:




$f(x)$




$f(x+a)$
$(a>0)$




$f(x-a)$
$(a>0)$

tabular:

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | 4 | 2 | 0 | 2 | 5 |

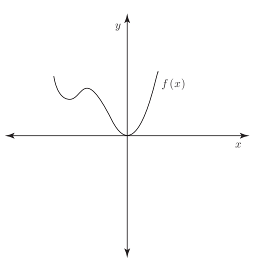
| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x+a)$ | | | | | |

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x-a)$ | | | | | |

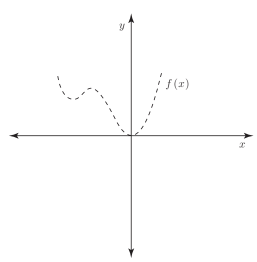
(e.g. $a=2$)

**Reflections** (vertical) (horizontal)

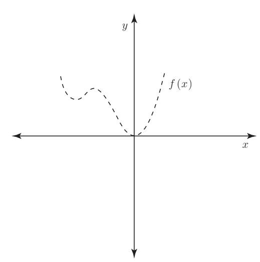
graphic:




$f(x)$




$-f(x)$




$f(-x)$

tabular:

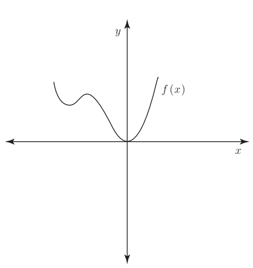
| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | 4 | 2 | 0 | 2 | 5 |

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $-f(x)$ | | | | | |

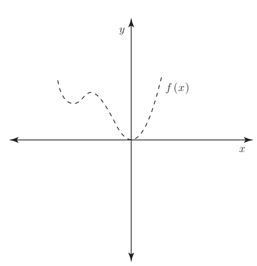
| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(-x)$ | | | | | |

**Stretches/Compressions** (vertical)

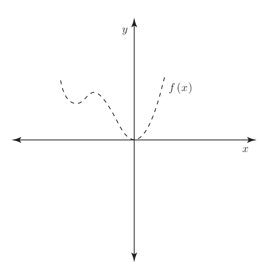
graphic:




$f(x)$




$af(x)$
$(a>1)$




$af(x)$
$(0<a<1)$

tabular:

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | 4 | 2 | 0 | 2 | 5 |

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $af(x)$ | | | | | |

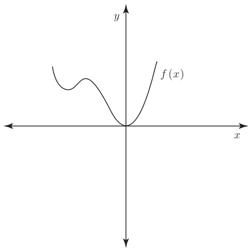
(e.g. $a=2$)

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $af(x)$ | | | | | |

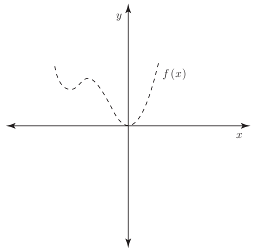
(e.g. $a=\frac{1}{2}$)

**Stretches/Compressions** (horizontal)

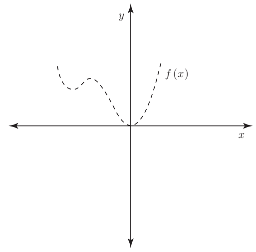
graphic:




$f(x)$




$f(ax)$
$(a > 1)$




$f(ax)$
$(0 < a < 1)$

tabular:

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | 4 | 2 | 0 | 2 | 5 |

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(ax)$ | | | | | |

(e.g. $a = 2$)

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(ax)$ | | | | | |

(e.g. $a = \frac{1}{2}$)

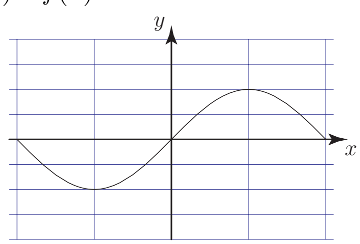
In each of the preceding cases, discuss the effects of the shift or stretch on the average rate of change of the function on the interval $[-2, 2]$.

## 0.2 Odd & Even Functions

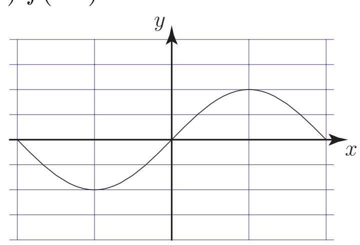
Sketch the following functions.

1. Sketch the indicated reflection over the graph of $g(x)$ in each exercise.

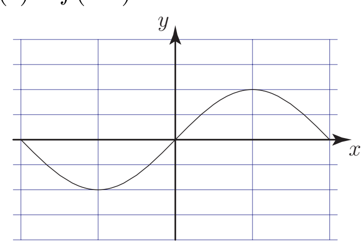
(a) $-f(x)$




(b) $f(-x)$

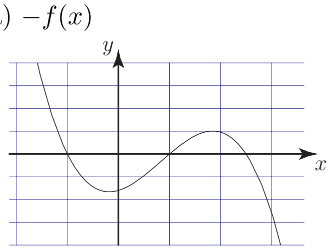



(c) $-f(-x)$

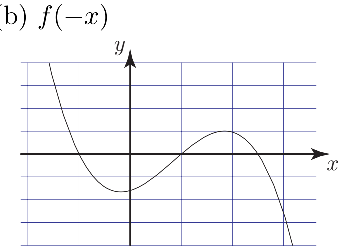



2. Sketch the indicated function over the graph of $f(x)$ in each exercise.

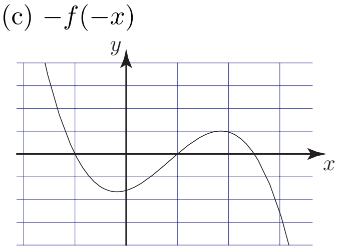
(a) $-f(x)$




(b) $f(-x)$

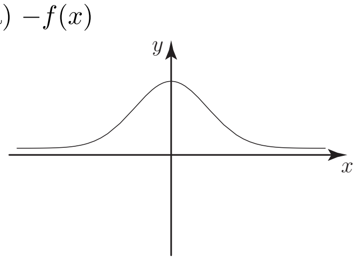



(c) $-f(-x)$

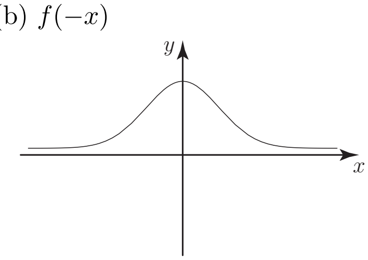



3. Sketch the indicated function over the graph of $f(x)$ in each exercise.

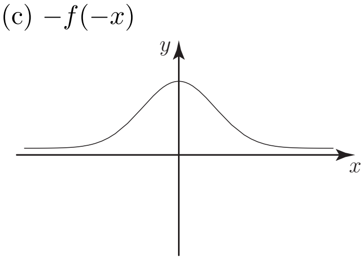
(a) $-f(x)$




(b) $f(-x)$




(c) $-f(-x)$



Let's remember the definitions of odd and even functions:

**Definition 0.1** *Odd and Even*

*A function, $f(x)$, is even if $f(-x) = f(x)$ for any $x$ in its domain.*
*A function, $f(x)$, is odd if $f(-x) = -f(x)$ (or equivalently $f(x) = -f(-x)$) for any $x$ in its domain.*

Which of the functions above are even? odd? neither? Explain.

Do odd and even functions behave like odd and even numbers? (How far does the metaphor extend?)
For example, we know the sum of two even numbers is an even number, is the sum of two even functions an even function? What about two odd functions? An odd and an even? What about the rules for products?

Begin your inquiry by choosing some examples and playing around with them. Remember that in order to disprove something you only need a single example showing it's not true. However, in order to *prove* something, you need to work in general terms using definitions rather than specific cases to show it works for *all* cases.

**Example 0.1**

The sum of an even function and an even function is an even function:
We want to show that if $f$ and $g$ are even functions, then their sum is an even function.
Let $h(x) = f(x) + g(x)$. We need to show that $h(-x) = h(x)$.
$h(-x) = f(-x) + g(-x)$
$= f(x) + g(x)$ Since both $f$ and $g$ are even.
$= h(x)$. Notice that we started with $h(-x)$ and concluded with $h(x)$.
Therefore the sum of two even functions is even.

**Questions 0.1**

1. Sums. Give either a proof or a counter example for each.
   (a) Is the sum of an odd function and an odd function even? Odd? Neither?
   (b) Is the sum of an odd function and an even function odd? Even? Neither?

2. Products. Give either a proof or a counter example for each.
   (a) Is the product of an even function and an even function even? Odd? Neither?
   (b) Is the product of an odd function and an odd function odd? Even? Neither?
   (c) Is the product of an odd function and an even function even? Odd? Neither?

## 0.3 Compositions

1. If $h(x) = f(g(x))$, decompose each function into functions $f$ and $g$ where $f(x) \neq x$ and $g(x) \neq x$.

(a) $h(x) = \sin(x^2 + 1)$
$f(x) =$ __________ $g(x) =$ __________

(b) $h(x) = (3x - 2)^3$
$f(x) =$ __________ $g(x) =$ __________

(c) $h(x) = \frac{1}{x^2-4}$
$f(x) =$ __________ $g(x) =$ __________

(d) $h(x) = e^{-x^2}$
$f(x) =$ __________ $g(x) =$ __________

(e) $h(x) = \frac{1}{\sqrt{3-x}}$
$f(x) =$ __________ $g(x) =$ __________

(f) $h(x) = \frac{1}{x^2} - 4$
$f(x) =$ __________ $g(x) =$ __________

2. What are the domains of the following functions?

(a) $a(x) = \ln x$ (b) $b(x) = \frac{1}{x^2+1}$ (c) $c(x) = a(b(x))$ (d) $d(x) = b(a(x))$

3. (a) On what interval in $[-\pi, \pi]$ is the function $s(x) = x^2$ increasing? On what interval is it decreasing?

(b) On what interval(s) in $[-\pi, \pi]$ is the function $t(x) = \sin x$ increasing? Decreasing?

(c) On what interval(s) in $[-\pi, \pi]$ is the function $w(x) = t(s(x))$ increasing? Decreasing?

4. The tables of $m(x)$ and $p(x)$ are given below.

| $x$ | $-6$ | $-4$ | $-2$ | $0$ | $2$ | $4$ | $6$ |
|---|---|---|---|---|---|---|---|
| $m(x)$ | $-1$ | $5$ | $8$ | $7$ | $3$ | $-2$ | $-11$ |

| $x$ | $-7$ | $-1$ | $5$ | $8$ | $7$ | $3$ | $-2$ |
|---|---|---|---|---|---|---|---|
| $p(x)$ | $-6$ | $-4$ | $-2$ | $0$ | $2$ | $4$ | $6$ |

If $u(x) = p(m(x))$, find . . .

(a) $u(0) =$ ____________________

(b) $u(2) =$ ____________________

If $w(x) = m(p(x))$, find . . .

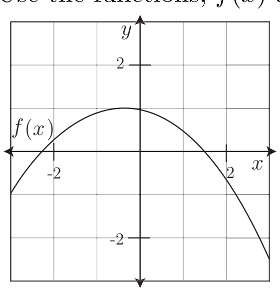
(c) $w(-1) =$ ____________________

(d) $w(2) =$ ____________________

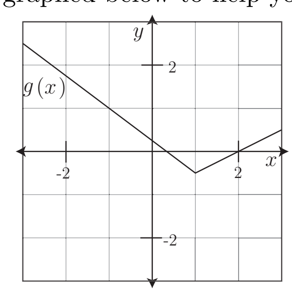
(e) For what value of $x$ is $w(x) = 7$?

5. Suppose $f(x)$ is an even function and $g(x)$ is an odd function. If $h(x) = f(g(x))$ and $k(x) = g(f(x))$, comment on whether they are even, odd, or neither.

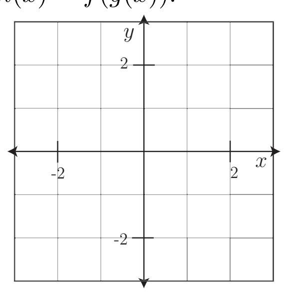
6. Use the functions, $f(x)$ and $g(x)$ graphed below to help you sketch $h(x) = f(g(x))$.




$f(x)$

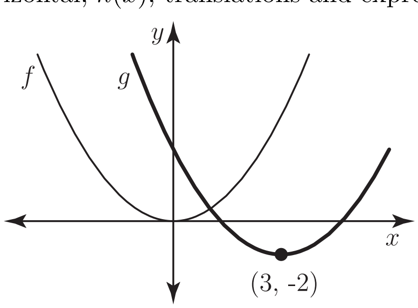



$g(x)$




$h(x)$

7. $g(x)$ is a translation of the parabola $f(x)$ as shown below. Write functions for vertical, $v(x)$, and horizontal, $h(x)$, translations and express $g(x)$ as a composition in terms of $f$, $h$, and $v$.



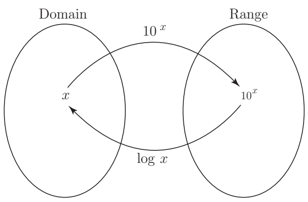

$h(x) =$ ____________________

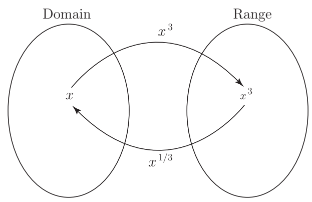
$v(x) =$ ____________________

$g(x) =$ ____________________

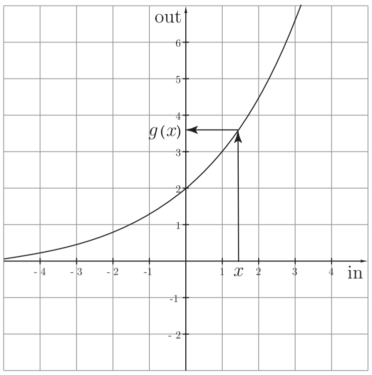
## 0.4 Inverses

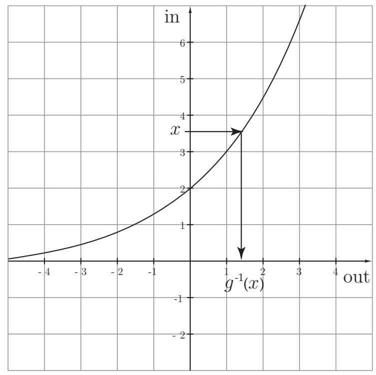
If a function maps values in its domain to new values in its range, then its inverse reverses the process by mapping values in the range (of the original function) back to its domain. (see below).




Equivalently,




In order for a function to have an inverse it isn't enough that we define a mapping reversing the original process. In order to actually reverse the original function, we must have some guarantee that we will get back to where we started from, that is, the inverse must also be a function. For this reason, $f(x) = x^2$ , $x \in \mathbb{R}$ does not have an inverse (since $f^{-1}(x) = \pm\sqrt{x}$ and $f^{-1}(9)$ could map to either $-3$ or 3, for example.), however, $f(x) = x^2$ , $x \in \mathbb{R}^+$ does have an inverse, namely $f^{-1}(x) = \sqrt{x}$.

**1.** Are the functions describing these situations invertible?

(a) The temperature of an oven as a function of the time since it was turned on.

(b) The number of people on a bus as a function of the time of day.

**2.** For $g(x) = 10^x - x$, find $g^{-1}(997)$.

**3.** For each function below, identify a proper domain and give the inverse function.

a) $f(x) = e^{1-3x}$ b) $g(x) = x^2 - 1$ c) $h(x) = \dfrac{x}{x+1}$

**4.** The following table shows values for the function $f$.

| $x$ | $-6$ | $-3$ | $-1$ | 0 | 1 | 4 | 7 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-6$ | $-5$ | 0 | 2 | 4 | 7 | 9 |

(a) Find $f(4)$.

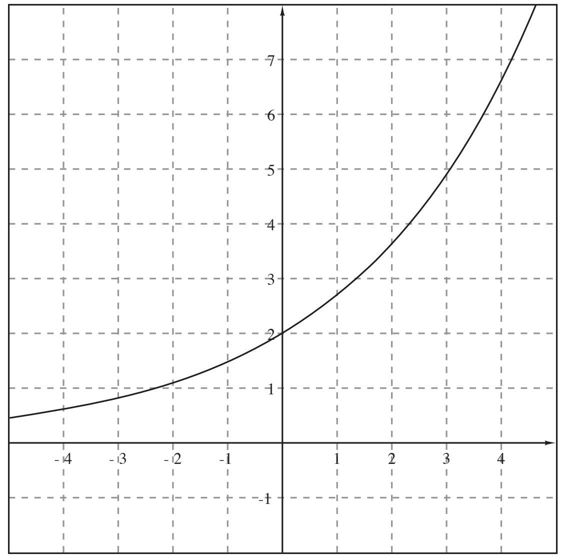
(b) Solve $f(x) = 4$.

(c) Solve $f(x) = x$.

(d) Find $f^{-1}(9)$.

(e) Solve $f^{-1}(x) = 0$.

5. Use the exponential function, $g(x)$, graphed below to answer the given questions.




(a) Estimate $g(0)$.

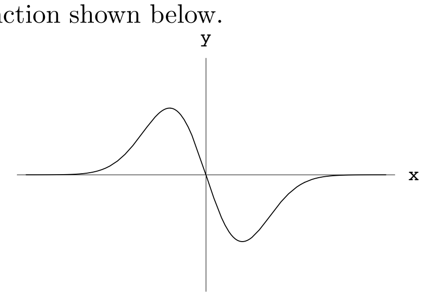
(b) Estimate $g(2)$.

(c) Estimate $g^{-1}(1)$.

(d) Estimate $g^{-1}(5)$.

(e) Estimate $g^{-1}(5.5)$.

6. On the same set of axes, sketch the inverse of the function shown below.




7. If $g(x) = 2^x + \sqrt{x} - 1$, find $g^{-1}(17)$.

8. Suppose $P = f(t)$ gives the population of a city in millions of people as a function of time in years since 1950.
(a) Give the units and interpret the meaning of $f(30) = 9.2$.

(b) Give the units and interpret the meaning of $f^{-1}(15) = 49$.