0.1 Limit Properties

The list below includes common properties of limits.

(1) $\lim_{x\to a}(f(x)+g(x)) = f(a)+g(a)$

(2) $\lim_{x\to a}(f(x)\cdot g(x)) = f(a)\cdot g(a)$

(3) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{f(a)}{g(a)}$

(4) $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x)$, $k\in\mathbb{R}$

(5) $\lim_{x\to a} k = k$

(6) $\lim_{x\to a} x^n = a^n$, $n\in\mathbb{N}$

0.1.1 Limits at Infinity

(1) $\lim_{x\to\infty} x = \infty, \quad \lim_{x\to\infty} e^x = \infty$

(2) $\lim_{x\to\infty}\frac{1}{x} = 0, \quad \lim_{x\to\infty} e^{-x} = \lim_{x\to\infty}\frac{1}{e^x} = 0$

(3) $\lim_{x\to a} x^n = a^n$, $n\in\mathbb{N}$

0.1.2 Infinite Limits

(1) $\lim_{x\to 0^+}\frac{1}{x} = \infty, \quad \lim_{x\to 0^-}\frac{1}{x} = -\infty$ So $\lim_{x\to 0}\frac{1}{x}$ is undefined.

0.2 Limit at a Finite Value

Example: Find $\lim_{x\to 3}\frac{x^3-x^2+x-21}{x^2-x-6}$

First check to see if direct substitution produces the indeterminate form $\frac{0}{0}$.

(If not, the form $\frac{n}{0}$, $(n\neq 0)$ goes to $\pm\infty$ while the form $\frac{0}{n}$, $(n\neq 0)$ goes to 0.)

Since $\lim_{x\to 3}\frac{x^3-x^2+x-21}{x^2-x-6}$ has the form $\frac{0}{0}$, we proceed by trying to factor and simplify:

$$\lim_{x\to 3}\frac{x^3-x^2+x-21}{x^2-x-6} = \lim_{x\to 3}\frac{x^3-x^2+x-21}{(x-3)(x+2)} \longrightarrow \text{(factor top by long division)} \longrightarrow \lim_{x\to 3}\frac{(x-3)(x^2+2x+7)}{(x-3)(x+2)}$$

$$\lim_{x\to 3}\frac{\cancel{(x-3)}(x^2+2x+7)}{\cancel{(x-3)}(x+2)} = \lim_{x\to 3}\frac{x^2+2x+7}{x+2} = \frac{(3)^2+2(3)+7}{(3)+2} = \frac{22}{5} \qquad \square$$

0.3 Limit at Infinity

Example: Find $\lim_{x\to\infty}\frac{x^3-2x^2+3x-1}{3x^3+x}$

Begin by dividing each term by the largest power among the terms (x^3):

$$\lim_{x\to\infty}\frac{x^3-2x^2+3x-1}{3x^3+x} = \lim_{x\to\infty}\frac{\frac{x^3}{x^3}-\frac{2x^2}{x^3}+\frac{3x}{x^3}-\frac{1}{x^3}}{\frac{3x^3}{x^3}+\frac{x}{x^3}} = \frac{\lim_{x\to\infty}\frac{x^3}{x^3}-\lim_{x\to\infty}\frac{2x^2}{x^3}+\lim_{x\to\infty}\frac{3x}{x^3}-\lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty}\frac{3x^3}{x^3}+\lim_{x\to\infty}\frac{x}{x^3}}$$

$$= \frac{1-0+0-0}{3+0} = \frac{1}{3} \qquad \square$$

Equivalently, this amounts to a shortcut. Since x^3 grows faster than x^2 or x, for big enough values of x, $\frac{x^3-2x^2+3x-1}{3x^3+x} \approx \frac{x^3}{3x^3}$. So $\lim_{x\to\infty}\frac{x^3-2x^2+3x-1}{3x^3+x} = \lim_{x\to\infty}\frac{x^3}{3x^3} = \lim_{x\to\infty}\frac{\cancel{x^3}}{3\cancel{x^3}} = \frac{1}{3} \qquad \square$

0.4 Horizontal Asymptotes

Horizontal asymptotes are limiting values of a function over a long period (positive or negative). Consequently, they're found by taking the infinite limits $y = \lim_{x\to\infty} f(x)$ and $y = \lim_{x\to-\infty} f(x)$.

Example: The horizontal asymptotes of $f(x) = \frac{2e^x+3e^{-x}}{3e^x+2e^{-x}}$ are given by,

$$y = \lim_{x\to\infty}\frac{2e^x+3e^{-x}}{3e^x+2e^{-x}} \longrightarrow \frac{2e^x+\cancelto{0}{3e^{-x}}}{3e^x+\cancelto{0}{2e^{-x}}} = \lim_{x\to\infty}\frac{2\cancel{e^x}}{3\cancel{e^x}} = \frac{2}{3} \quad \text{and} \quad y = \lim_{x\to-\infty}\frac{2e^x+3e^{-x}}{3e^x+2e^{-x}} = \lim_{x\to\infty}\frac{2e^{-x}+3e^x}{3e^{-x}+2e^x} = \frac{3}{2} \qquad \square$$